Existence and Dimension of the Attractor for the Bénard Problem on Channel-Like Domains

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Abstract

The Bénard problem, a system with the Navier-Stokes equations for the velocity field coupled with a convection-diffusion equation for the temperature is considered. Non-homogeneous boundary conditions, external force and heat source in dual function spaces, and an arbitrary spatial domain (possibly nonsmooth and unbounded) as long as the Poincaré inequality holds on it (channel-like domain) are allowed. Moreover our approach, unlike in previous works, avoids the use of the maximum principle which would be problematic in this context. The mathematical formulation of the problem, the existence of global solution and the existence and finite dimensionality of the global attractor are proved.

1 Introduction

The Bénard problem is a system with the Navier-Stokes equations for the velocity field coupled with a convection-diffusion equation for the temperature. The velocity influences the diffusion of the temperature and the temperature induces a convective motion of the fluid via the Boussinesq approximation. The latter effect is due to imbalances between the hydrostatic pressure and the gravitational force generated by the variation of density of the fluid caused by the temperature (see [2] and [12]). Basically, when the temperature of a fluid parcel in hydrostatic equilibrium is increased, this fluid parcel expands and the buoyant force acting on it increases, resulting in an upward net force.

We are allowing for the following new features compared with previous works:

a) non-homogeneous boundary conditions and non-homogeneous volume forces and heat source at the same time;

b) external force and heat source in dual function spaces, hence less smooth; and

c) the spatial domain is an arbitrary (possibly nonsmooth and/or unbounded) open set as long as the Poincaré inequality holds on it (channel-like domain).

The usual way to handle non-homogeneous boundary conditions is to homogenize them by subtracting from the velocity field a suitable "background flow." This background flow can be constructed via a Hopf-type technique which extends the given boundary values to a suitable velocity field defined on the whole domain. This construction works for bounded smooth ($C^2$) domains (see [8] and [22]) and also for Lipschitz domains with zero flux through the boundary (see [3]). However, in some particular nonsmooth (see Section 5.3) and/or unbounded (see Section 5.4) domains with nonzero flux through the boundary it can still be constructed by an adaptation of Hopf's technique. For this reason we assume at first that we are already given a suitable background flow defined on the whole domain and then the construction is left to each application.

We prove the existence of global solutions and of the global attractor in the usual $L^2$-like space. We also prove that the global attractor has finite Hausdorff and fractal dimensions and we estimate its dimension in terms of suitable Reynolds, Prandtl, Grashof and Rayleigh numbers.

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1 INTRODUCTION

This work is an extension of Rosa [17], where the Navier-Stokes equations were studied on unbounded, nonsmooth domains in which the Poincaré inequality is valid and for general forces in dual function space, but with homogeneous boundary conditions and without temperature influences.

The usual approach to prove the existence of the global attractor is to obtain an absorbing set in a more regular space and then use Sobolev's compact embedding. For a compact embedding it is enough to assume that the domain is bounded. But to obtain this absorbing set on nonsmooth domains, it is a bit involved (see [11]) and it would not be enough for certain unbounded domains (like a channel). For this reason, we follow the approach in Rosa [17], using the energy-equation method. In doing so we treat, for instance, the case of a flow in a channel with irregularities and a cavity (see Figure 1), an infinite channel with an irregular obstacle (see Figure 2) and a multi-channel (see Figure 3).

Two earlier references for the mathematical study of the Bénard problem are the articles by Foias, Manley, and Temam [3] (see also [20, section III 3.5. and VI 4.5]) and by Manley, Marion, and Temam [14]. In [5] the existence of the global attractor for the Bénard problem is studied in a square with many different boundary conditions (Dirichlet, periodic, free, and combinations of these in different directions) but without external heat source. In [14] the authors study a system modelling combustion in a rectangle, which includes the Bénard problem, with non-zero external heat source, but with a particular type of boundary conditions for the temperature, namely homogeneous Neumann condition (normal derivative is zero) in all sides except in one, and homogeneous Dirichlet condition (temperature is zero) in the remaining one.

In the Bénard problem the temperature appears in the momentum equation as a driving force. When the temperature is constant throughout the boundary, which is essentially the situation in [14], we have, after subtracting a background flow and a background temperature, a partial coupling of the system. The velocity term is eliminated from the temperature equation in the a priori estimates due to the well-known orthogonality property of the bilinear term. In this way, the operator appearing in the temperature equation is coercive. This yields an $L^2$ bound on the temperature which is used in the momentum equation to bound the term from the Boussinesq approximation. This implies the existence of a global solution and of an absorbing ball.

When the temperature is not constant throughout the boundary, which is the case in [5], we have, after subtracting a background flow and a background temperature, a full coupling of the system. It was suspected in [20] that the resulting operator is not coercive unless the viscosity and thermometric conductivity coefficients are relatively small. This coercivity is a fundamental step for the well-posedness and for the proof of the existence of the global attractor.

The full coupling of the system was bypassed in [5] by proving a maximum principle for the original temperature equation, before the subtraction of the background flow and temperature. This maximum principle yields an $L^\infty$ bound on the temperature which is used in the momentum equation to bound the forcing term due to the temperature.

A maximum principle is also proved in [14], but it is not crucial for the existence of the global attractor, neither for the finite dimensionality of the attractor. It is used to yield a more physically relevant estimate of the dimension of the attractor.

There are three problems with these approaches in our case:

a) A maximum principle for the temperature equation was proved in [5] in the absence of external heat source. For non-zero external heat source, a maximum principle was proved in [14], but the proof is delicate and technically involved and it was applied only for a simple set of boundary conditions

b) For unbounded domains it is not enough to have an $L^\infty$ bound on the temperature because this does not yield an $L^2$ bound on the temperature as a forcing term in the momentum equation.

c) Even for bounded domains, we would need to assume that the initial data and external heat source are in $L^\infty$, in contrast with our approach which allow for general external heat source (in dual function space) and initial data in $L^2$. An interesting situation is that of a Dirac-like heat source.

As far as we know, this is the first work that treats the Bénard problem in unbounded domains. Even on bounded domains, this is the first time that non-trivial boundary conditions and an external heat source are considered at the same time.

We avoid altogether using a maximum principle by treating the momentum and temperature equations as a system. We define an operator on the velocity field and temperature which is indeed coercive. The difficulty for
the coercivity comes from the presence of two types of terms: that from the Boussinesq approximation and those from the background flow and background temperature. The background terms are taken care of by assumptions based on the Hopf-type technique. The Boussinesq term is handled by a suitable choice of norm on the product space of velocity field × temperature. We exploit then an energy equation valid for this Benard system to obtain the so-called asymptotic compactness of the semigroup. This implies (see [20]) the existence of the global attractor.

This article is organized as follows: In Section 2 we build the mathematical framework necessary to set up the problem, showing existence and a weak continuity result. In Section 3 we prove the existence of the global attractor. In Section 4 we estimate the dimension of the global attractor. Finally, in Section 5, we illustrate our results on some particular geometries and boundary conditions relevant in applications.

2 Mathematical Setting and Existence of Solutions

2.1 Formulation of the Model

We consider an incompressible viscous fluid of constant density in a region $\Omega \subset \mathbb{R}^2$ governed by the Navier-Stokes equations. The fluid is subjected to a buoyant effect due to the variation of density of the fluid caused by the temperature that is modeled using the Boussinesq approximation (see [2]). The temperature satisfies a heat equation, with convection given by the velocity of the fluid. The momentum and temperature equations may be subjected to external force and heat source respectively.

The variables are the velocity $u(x,t)$, the pressure $p(x,t)$ and the temperature $T(x,t)$ of the fluid at the point $x \in \Omega$ and at time $t \geq 0$, which are determined by the following equations:

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f_u + \sigma \varepsilon_2 (T - T_r), \quad (1) \]

\[ \frac{\partial T}{\partial t} + (u \cdot \nabla) T - \kappa \Delta T = f_T, \quad (2) \]

\[ \nabla \cdot u = 0. \quad (3) \]

The constants $\nu > 0$ and $\kappa > 0$ are the coefficients of viscosity and thermometric conductivity, respectively. The buoyant effect due to the Boussinesq approximation is given through the constant $\sigma = \alpha g$, where $g$ is the gravitation acceleration, and $\alpha$ is the volume-expansion coefficient of the fluid. We also have $\varepsilon_2$, the unit vector on the vertical direction, and $T_r$, a reference temperature of the medium.

The given functions are $f_u(x)$, the external body force, and $f_T(x)$, the external heat source.

The boundary conditions for equations (1)-(3) are given by

\[ \begin{cases} 
\quad u(x,t) = \phi_u(x) & \text{for all } x \in \partial \Omega, \\
\quad T(x,t) = \phi_T(x) & \text{for all } x \in \partial \Omega, 
\end{cases} \quad (4) \]

where $\phi_u$ and $\phi_T$ are the velocity and the temperature, respectively, given on the boundary.

The initial conditions are given by

\[ u(\cdot,0) = u_0, \quad T(\cdot,0) = T_0. \]

The domain $\Omega$ can be an arbitrary (bounded or unbounded) open set in $\mathbb{R}^2$ without any regularity assumption on its boundary $\partial \Omega$. The only assumption is that the Poincaré inequality holds on it, i.e., there exist $\lambda_1 > 0$ such that

\[ \int_{\Omega} \phi^2 dx \leq \frac{1}{\lambda_1} \int_{\Omega} |\nabla \phi|^2 dx, \quad \forall \phi \in H^1_0(\Omega). \quad (5) \]
2.2 Reformulation of the Model

We reformulate the problem to make sense out of the boundary condition (4).

If $\partial \Omega$ is regular enough and bounded then given a smooth function $\phi$ on the boundary we can find $\Phi$, a "background flow," in $\Omega$ with its trace on $\partial \Omega$ equal to $\phi$ (see Sections 5.1 and 5.2). Since we assume neither regularity nor boundedness, we need to construct, for each application (see Sections 5.3 and 5.4), $u_b$ and $T_b$ defined in $\Omega$ (the background flow and temperature) such that $\phi_b = u_b$ and $\phi_T = T_b$ on $\partial \Omega$ in some sense, satisfying the conditions given in Section 2.5.

We consider the boundary condition to be achieved in the sense that $u - u_b$ and $T - T_b$ belong to Sobolev spaces obtained as closures of compactly supported smooth functions. Then we define $v = u - u_b$ and $\theta = T - T_b$ and we can finally rewrite (1)-(3) as

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu \Delta v + \nabla p = f_u -(u_b \cdot \nabla)v - (v \cdot \nabla)u_b + \sigma \tilde{v}^2 \theta,$$

$$\frac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta - \kappa \Delta \theta = f_T - (u_b \cdot \nabla)\theta - (v \cdot \nabla)T_b,$$

$$\nabla \cdot v = 0,$$

where $f_u$ and $f_T$ are defined by

$$f_u = f_u + \nu \Delta u_b - (u_b \cdot \nabla)u_b + \sigma \tilde{v}^2(T_b - T_f),$$

$$f_T = f_T + \kappa \Delta T_b - (u_b \cdot \nabla)T_b.$$  

The boundary conditions for equations (6)-(8) are given by

$$v(x, t) = 0, \quad \theta(x, t) = 0, \quad \text{for all } x \in \partial \Omega.$$  

The initial conditions are given by

$$v(\cdot, 0) = v_0 = u_0 - u_b, \quad \theta(\cdot, 0) = \theta_0 = T_0 - T_b.$$  

2.3 Function Spaces

In this Section we construct several function spaces necessary to write problem (6)-(11) in its variational formulation, with the unknown function $z = (v, \theta)$. Let

$$\mathcal{V} = \{v \in (L^2(\Omega))^2; \quad \nabla \cdot v = 0\}.$$  

Let us introduce the notation

$$L^2(\Omega) = (L^2(\Omega))^2, \quad \mathcal{H}(\Omega) = (H^1(\Omega))^2, \quad \mathcal{H}_0(\Omega) = (H_0^1(\Omega))^2.$$  

We define

$$V_1 = \text{closure of } \mathcal{V} \text{ in the } \mathcal{H}_0^1(\Omega) \text{ norm}, \quad V_2 = H_0^1(\Omega),$$

$$H_1 = \text{closure of } \mathcal{V} \text{ in the } L^2(\Omega) \text{ norm}, \quad H_2 = L^2(\Omega),$$

$$V = V_1 \times V_2, \quad H = H_1 \times H_2.$$  

The inner product and norm in $V_1$ are given by

$$\langle (v, \bar{v}) \rangle = \sum_{i,j=1}^2 \int_{\Omega} \nabla v_i \cdot \nabla \bar{v}_i \, dx, \quad \forall v, \bar{v} \in V_1,$$

$$\|v\| = \langle (v, v) \rangle^{1/2}, \quad \forall v \in V_1.$$  

Due to (5) this norm is equivalent to the usual one in $\mathcal{H}_0^1(\Omega)$. For simplicity we use the same notations $\| \cdot \|$, $\langle \cdot, \cdot \rangle$ to denote the inner product and norm in $V_2$, given by

$$\langle (\theta, \bar{\theta}) \rangle = \int_{\Omega} \nabla \theta \cdot \nabla \bar{\theta} \, dx, \quad \forall \theta, \bar{\theta} \in V_2,$$

$$\|\theta\| = \langle (\theta, \theta) \rangle^{1/2}, \quad \forall \theta \in V_2.$$  

2 Mathematical Setting and Existence of Solutions

Repeating the notation again for simplicity, we define the inner product and norm in $V$ by

$$
\langle (z, \tilde{z}) \rangle = \langle (v, \bar{v}) \rangle + \gamma \langle (\theta, \bar{\theta}) \rangle, \quad \forall z = (v, \theta), \tilde{z} = (\bar{v}, \bar{\theta}) \in V,
$$

$$
|z| = \langle (z, \tilde{z}) \rangle^{1/2}, \quad \forall z \in V,
$$

where $\gamma$ is defined so that it satisfies

$$
\gamma \geq \left( \frac{\sigma}{\lambda_1} \right)^2 \frac{1}{\nu \kappa}.
$$

This constant $\gamma$ is chosen so that an operator to be defined later (related to the linear part of the system of equations) is coercive under the norm defined above. Moreover, the choice of $\gamma$ above makes the two terms in the definition of the inner product in $V$ dimensionally consistent in physical units.

The norm and inner product in $H_1$ and $H_2$ are the usual ones inherited from $L^2(\Omega)$ and $L^2(\Omega)$ respectively. We define the inner product and norm in $H$ by

$$
\langle (z, \tilde{z}) \rangle = \langle (v, \bar{v}) \rangle + \gamma \langle (\theta, \bar{\theta}) \rangle, \quad \forall z = (v, \theta), \tilde{z} = (\bar{v}, \bar{\theta}) \in H,
$$

$$
|z| = \langle (z, \tilde{z}) \rangle^{1/2}, \quad \forall z \in H.
$$

Since $\gamma$ is positive, the norms and inner products defined above for $H$ and $V$ are equivalent to the usual ones defined on these product spaces.

It follows from (5) that for all $v \in V_1$ and $\theta \in V_2$ we have

$$
|v|^2 \leq \frac{1}{\lambda_1} ||v||^2, \quad |\theta|^2 \leq \frac{1}{\lambda_1} ||\theta||^2.
$$

Applying Riesz’s representation theorem, we can identify the dual space $H'$ with $H$, obtaining the following relation:

$$
V \subset H = H' \subset V',
$$

where the injections are continuous and each space is dense in the following ones.

We assume that $f_u \in V_1'$, $f_T \in V_2'$ and $T_r \in V_2'$. Let us consider $F = (f_u, f_T) \in V'$. It is easy to deduce that

$$
\langle F, z \rangle_{V', V} = \langle f_u, v \rangle_{V_1, V_1} + \gamma \langle f_T, \theta \rangle_{V_2, V_2}.
$$

2.4 Some Operators for the Variational Form

Let us define the bilinear forms $a_i : V_1 \times V_1 \rightarrow \mathbb{R}$, for $i = 1 \ldots 2$, $a : V \times V \rightarrow \mathbb{R}$, and its corresponding linear operator $A : V \rightarrow V'$, by

$$
a_i (v, \bar{v}) = \langle (v, \bar{v}) \rangle = \int_{\Omega} \sum_{i,j=1}^{2} \nabla v_j \cdot \nabla \bar{v}_i \, dx,
$$

$$
a_2 (\theta, \bar{\theta}) = \langle (\theta, \bar{\theta}) \rangle = \int_{\Omega} \nabla \theta \cdot \nabla \bar{\theta} \, dx,
$$

$$
a(z, \tilde{z}) = \langle A z, \tilde{z} \rangle_{V', V} = \nu a_i (v, \bar{v}) + \gamma \kappa a_2 (\theta, \bar{\theta}).
$$

The operator $A$ is clearly a homomorphism from $V$ into $V'$ and the bilinear form $a$ is coercive since

$$
\min (\nu, \kappa) ||z||^2 \leq a(z, z) = \langle A z, z \rangle_{V', V} \leq \max (\nu, \kappa) ||z||^2.
$$

Let us define $b_i : V_1 \times V_1 \times V_1 \rightarrow \mathbb{R}$, for $i = 1 \ldots 2$, $b : V \times V \rightarrow \mathbb{R}$, and $B(z) = B(z, z)$, its associated bilinear operator $B : V \times V \rightarrow V'$, by

$$
b_i (v, \bar{v}, \bar{\bar{v}}) = \int_{\Omega} \sum_{i,j=1}^{2} v_i \frac{\partial \bar{v}_j}{\partial x_i} \bar{\bar{v}}_j \, dx,
$$

$$
b (w, \bar{w}) = \int_{\Omega} \nabla w \cdot \nabla \bar{w} \, dx,
$$

$$
B (z, \bar{z}) = \int_{\Omega} \nabla z \cdot \nabla \bar{z} \, dx.
$$
\[ b_2(v, \hat{\theta}, \hat{\theta}) = \int_{\Omega} \sum_{i=1}^{2} v_i \frac{\partial \hat{\theta}}{\partial x_i} \, dx, \]
\[ b(z, \hat{z}, \hat{z}) = \langle B(z, \hat{z}), \hat{z} \rangle = b_1(v, \hat{v}, \hat{v}) + \gamma b_2(v, \hat{\theta}, \hat{\theta}). \]

We need the following lemma, which we quote from [11], to look into the properties of this operator.

**Lemma 1 (Ladyzhenskaya’s inequality)** If \( n = 2 \), for any open set \( \Omega \),
\[
\|\phi\|_{L^4(\Omega)} \leq \left( \frac{1}{2\lambda_1} \right)^{\frac{1}{2}} \|\nabla \phi\|_{L^2(\Omega)}, \quad \forall \phi \in H^1_0(\Omega). \tag{17}
\]

Using Lemma 1 and the Poincaré inequality (5) we obtain
\[
\|\phi\|_{L^4} \leq \left( \frac{1}{2\lambda_1} \right)^{\frac{1}{2}} \|\nabla \phi\|_{L^2(\Omega)}, \quad \forall \phi \in H^1_0(\Omega). \tag{18}
\]

**Lemma 2** For any open set \( \Omega \subset \mathbb{R}^2 \) and \( z, \tilde{z} \in V \) we have
\[ i) \|b(z, z, \tilde{z})\| \leq |z| \|z\| \|\tilde{z}\| \quad \text{and} \]
\[ ii) \|b(z, \tilde{z}, \tilde{z})\| = 0. \]

**Proof:** (sketch) It is obvious that
\[
1/\sqrt{2}(||v|| + \gamma^{1/2}||\tilde{\theta}||) \leq \sqrt{(||v||^2 + \gamma||\tilde{\theta}||^2)} = ||\tilde{z}||. \tag{19}
\]
We also have that
\[
|b_1(v, \hat{v}, \hat{v})| = \left| \int (v \cdot \nabla) \hat{\hat{v}} \cdot \hat{\hat{v}} \right| \leq \|v\|_{L^4} \|\nabla \hat{v}\|_{L^4} \|\hat{v}\|_{L^4}, \tag{20}
\]
and a similar inequality for \( b_2 \). From (20) and (17) we have
\[
|b_1(v, v, \tilde{v})| \leq 1/\sqrt{2} |v| \|v\||\tilde{v}| \leq 1/\sqrt{2} |z| \|z\||\tilde{v}| \]
and
\[
|b_2(v, \hat{\theta}, \hat{\theta})| \leq 1/\sqrt{2} |v| \|v\| |\hat{\theta}| \|\hat{\theta}\| \leq 1/\sqrt{2} |z| \|z\|^{\gamma^{-1}} \|\tilde{\theta}\|. \]

Hence
\[
|b| \leq |b_1| + \gamma |b_2| \leq 1/\sqrt{2} |z| \|z\| |\|\tilde{\theta}|| + \gamma^{1/2} |\tilde{\theta}||. \]

Using (19) we obtain i).
As for ii), we obtain it by integration by parts.

Applying Lemma 2 we see that \( B \) has the orthogonality property
\[
\langle B(z, z), z \rangle_{V^*, V} = 0 \tag{21}
\]
and satisfies
\[
\|B(z)\|_{V^*} \leq |z| \|z\|, \quad \forall z \in V. \tag{22}
\]
2.5 Hypotheses on $u_b$ and $T_b$

In order to make sense out of the terms involving $u_b$ and $T_b$ in (6) and (7), we need to make some hypotheses on $z_b = (u_b, T_b)$. As a general assumption, so that we can write the integrals below, we assume that

$$z_b \in L^1_{\text{loc}}(\Omega) \times L^1_{\text{loc}}(\Omega).$$  \hspace{1cm} (23)

We assume also that

$$\nabla \cdot u_b = 0$$

in the distribution sense. \hspace{1cm} (24)

Concerning $\Delta u_b$ and $T_b$ in (9), $\Delta T_b$ in (10), we assume that

$$\Delta u_b \in V'_1,$$  \hspace{1cm} (25)

$$\Delta T_b \in V'_2$$  \hspace{1cm} (26)

and

$$T_b \in V'_2.$$  \hspace{1cm} (27)

For $z = (v, \theta)$, $\hat{z} = (\hat{v}, \hat{\theta})$ and $\bar{z} = (\bar{v}, \bar{\theta})$, let us introduce the formal notation

$$\overline{b}(z, \hat{z}, \bar{z}) = - \int (v \cdot \nabla) \bar{v} \cdot \hat{\theta} - \gamma \int (v \cdot \nabla) \bar{\theta} \cdot \hat{\theta}.$$  

We assume that $u_b$ and $T_b$ are such that the integrals above are well-defined and that there exists a constant $c_b$ such that, for all $z, \bar{z} \in V$,

$$|\overline{b}(z_b, z, \bar{z})| + |\overline{b}(z_b, z_b, \bar{z})| \leq c_b \|z\| \|\bar{z}\|.$$  \hspace{1cm} (28)

We assume that there exists constants $c_{u_b}$ and $c_{u_T}$ such that for all $v \in V_1$ and $\theta \in V_2$,

$$\left| \int (u_b \cdot \nabla) v \cdot u_b \right| \leq c_{u_b} \|v\| \quad \text{and} \quad \left| \int (u_b \cdot \nabla) \theta \cdot T_b \right| \leq c_{u_T} \|\theta\|.$$  \hspace{1cm} (29)

These constants will be relevant to estimate the dimension of the attractor. This conditions is equivalent to assume that there exists $c_{b_0}$ such that

$$|\overline{b}(z_b, z, z)| \leq c_{b_0} \|z\|.$$  

**Remark:** With an abuse of notation, we can replace conditions (28) and (29) by the assumption that for all $z \in V$,

$$B(z_b, z), B(z, z_b) \quad \text{and} \quad B(z_b, z_b) \in V'.$$

From (23) and (24) we obtain

$$\overline{b}(z_b, z, z) = 0.$$  \hspace{1cm} (30)

Finally, to obtain the coercivity of an operator to be defined later, we assume that for all $v \in V_1$ and $\theta \in V_2$,

$$\left| \int (v \cdot \nabla) v \cdot u_b \right| \leq \frac{\nu}{4} \|v\|^2 \quad \text{and} \quad \left| \int (v \cdot \nabla) \theta \cdot T_b \right| \leq \frac{\beta}{4} \|v\| \|\theta\|,$$  \hspace{1cm} (31)

where $\beta$ is a positive constant such that

$$\beta \leq \left( \frac{\nu}{\gamma} \right)^{1/2} \leq \left( \frac{\nu_k \lambda_1}{\mathbf{\sigma}} \right) \leq \frac{\nu_k \lambda_1}{|\mathbf{\sigma}|}.$$  \hspace{1cm} (32)
2.6 Other Operators for the Variational Form

We define \( r : V \times V \to \mathbb{R} \) and the associated linear operator \( R : V \to V' \) by
\[
    r(z, \tilde{z}) = \langle Rz, \tilde{z} \rangle = \tilde{b}(z_0, z) + \overline{b}(z, z_0, \tilde{z}) - \sigma(\tilde{e}_2\theta, \tilde{v}) .
\]
A straightforward computation yields
\[
    |r(z, \tilde{z})| \leq \sqrt{\nu_K}||z|| ||\tilde{z}|| .
\]
Using this and (28) we obtain
\[
    |r(z, \tilde{z})| \leq c_r||z|| ||\tilde{z}|| \quad \text{and} \quad ||Rz||_{V'} \leq c_r||z||
\]
with \( c_r = \sqrt{\nu_K} + c_b \). Also, from (30) we obtain
\[
    r(z, z) = \tilde{b}(z, z_0, z) - \sigma(\tilde{e}_2\theta, v) .
\]
Assuming that \( F = (f_u, f_T) \in V', T_r \in V'_2 \), we define \( e : V \to \mathbb{R} \) by
\[
    e(z) = \langle f_u, v \rangle + \gamma \langle f_T, \theta \rangle = \langle F, z \rangle - \langle (\nu \Delta u_b, \kappa \Delta T_b), z \rangle - \tilde{b}(z_0, z_0, z) + \sigma(\tilde{e}_2(T_0 - T_r), v) .
\]
From (15) we have \( \Psi \in V' \) defined as
\[
    \Psi = F - (\nu \Delta u_b, \kappa \Delta T_b) - B(z_0, z_0) + (\sigma \tilde{e}_2(T_0 - T_r), 0)
\]
such that
\[
    e(z) = \langle \Psi, z \rangle_{V', V} .
\]
From (15), (25) and (26), it is straightforward to check that
\[
    ||(\nu \Delta u_b, \kappa \Delta T_b), z)|| \leq \sqrt{2} \left( ||\Delta u_b||_{V'_2} + \kappa \gamma^{1/2} ||\Delta T_b||_{V'_2} \right) ||z|| \quad \text{and}
\]
\[
    ||\sigma(\tilde{e}_2(T_0 - T_r), v)|| \leq ||T_0 - T_r||_{V'_2} ||v|| .
\]
Applying these inequalities and (29) we obtain
\[
    e(z) \leq c_e ||z|| \quad \text{and} \quad ||\Psi||_{V'} \leq c_e ,
\]
with
\[
    c_e = ||F||_{V'} + \sqrt{2} \left( ||\Delta u_b||_{V'_2} + \kappa \gamma^{1/2} ||\Delta T_b||_{V'_2} + c_{ab} + \gamma^{1/2} c_{\theta r} \right) + \gamma ||T_0 - T_r||_{V'_2} .
\]

2.7 Weak Formulation

We consider now the following weak formulation of (6)–(12).

**Problem 1** For \( z_0 \in H \) given, find \( z = (v, \theta) \) such that
\[
    z \in L^2(0, T; V) \cap L^\infty(0, T; H), \quad \forall T > 0 ,
\]
\[
    \frac{d}{dt}(z, \tilde{z}) + a(z, \tilde{z}) + r(z, \tilde{z}) + b(z, z, \tilde{z}) = e(z), \quad \forall \tilde{z} \in V, \forall T > 0 ,
\]
\[
    z|_{t=0} = z_0 .
\]
Equation (37) is equivalent to the functional equation in \( V' \)
\[
    z' + (A + R)z + B(z) = \Psi \quad \text{in} \ V', \quad \forall T > 0 .
\]
where \( z' = (dv/dt, d\theta/dt) \), and \( \Psi \) is given by (36).
2.8 V-Ellipticity Condition

In order to guarantee global solutions and the existence of an attractor, we assume that $A + R$ is V-elliptic. More precisely, we assume that there exists $\delta > 0$ such that

$$
\langle (A + R) z, z \rangle \geq \delta \left( ||v||^2 + \gamma \kappa ||\theta||^2 \right). \tag{39}
$$

This condition is equivalent to the following condition: there exists $\delta' > 0$ such that

$$
\langle (A + R) z, z \rangle \geq \delta'||z||^2. \tag{39}
$$

**Lemma 3** If $u_0$ and $T_0$ satisfy (31) and (32) then the operator $A + R$ satisfies (39) with $\delta = 1/8$.

**Proof:** Using Hölder’s inequality in (32)

$$
\gamma \int (v \cdot \nabla) \theta \cdot T_0 \leq \frac{\gamma \beta^2}{4} ||v|| ||\theta|| \leq \frac{\gamma \beta^2}{8} ||v||^2 + \frac{\gamma \kappa}{8} ||\theta||^2.
$$

From (33), $\frac{\gamma \beta^2}{8} \leq \nu$. Therefore

$$
\gamma \int (v \cdot \nabla) \theta \cdot T_0 \leq \frac{\nu}{8} ||v||^2 + \frac{\gamma \kappa}{8} ||\theta||^2. \tag{40}
$$

From the definition of $\overline{b}$ and putting together (31) and (40) we obtain

$$
\overline{b}(z, z_0, z) \leq \frac{3\nu}{8} ||v||^2 + \frac{\gamma \kappa}{8} ||\theta||^2. \tag{41}
$$

From (15), $|\sigma|/\lambda_1 \leq (\gamma \nu \kappa)^{1/2}$. Using this and the Poincaré inequality (14) we obtain

$$
|\sigma(\overline{e}_2 \theta, v)| \leq \frac{|\sigma|}{\lambda_1} ||\theta|| ||v|| \leq (\gamma \kappa)^{1/2} ||\theta|| ||v|| \nu^{1/2}
$$

(by Hölder’s inequality)

$$
\leq \frac{\nu ||v||^2 + \gamma \kappa ||\theta||^2}{2}.
$$

From (35), (41) and the inequality above we have

$$
|\langle R z, z \rangle| = |r(z, z)| \leq |\overline{b}(z, z_0, z)| + |\sigma(\overline{e}_2 \theta, v)| \leq \frac{7\nu}{8} ||v||^2 + \frac{5\gamma \kappa}{8} ||\theta||^2.
$$

Using the definition of $A$, the definition of $||z||$ and the inequality above we obtain

$$
\langle (A + R) z, z \rangle \geq \nu ||v||^2 + \gamma \kappa ||\theta||^2 - |\langle R z, z \rangle|
$$

$$
\geq \frac{\nu}{8} ||v||^2 + \frac{3\gamma \kappa}{8} ||\theta||^2
$$

$$
\geq \frac{1}{8} (\nu ||v||^2 + \gamma \kappa ||\theta||^2).
$$


2.9 Existence of Solution

**Theorem 1** Suppose we are given $\nu > 0$ and $\kappa > 0$, $f_0 \in V_1'$, $f_T \in V_2'$, $T_r \in V_2'$, $z_0 = (u_0, T_0)$ satisfying (23)–(29) and (39). Then for any $z_0 \in H$ there exists a unique $z \in L^\infty(\mathbb{R}^+; H) \cap L^2(0, T; V)$ for all $T > 0$ such that (37) (hence (38)) holds. Since $z \in L^2(0, T; V)$, equation (38) implies that $z' \in L^2(0, T; V')$ for all $T > 0$. Hence $z \in C(\mathbb{R}^+; H)$. Furthermore $z|_{t=0} = z_0$.\[\square\]
Proof: (sketch) From (39) we have that \((A + R)\) is V-elliptic. The existence of solutions for \(t \in (0, T)\) is then classical (it follows by applying for example Theorem 3.1 of [20, pag. 114]).

Now we determine an energy equation for the solution. We define a symmetric bilinear form \([, ]: V \times V \rightarrow \mathbb{R}\) by

\[
[z, \bar{z}] = \langle (A + R)z, \bar{z} \rangle - \frac{\zeta \lambda_1}{2} (z, \bar{z}), \quad \forall z, \bar{z} \in V,
\]

where \(\zeta\) is defined as

\[
\zeta = \delta \min(\nu, \kappa),
\]

and where \(\delta \) is given by (39), the V-ellipticity condition.

From (34) and the definition of \(A\) we have

\[
[z]^2 \equiv [z, z] \leq (\max(\nu, \kappa) + c_r) ||z||^2.
\]

Let \(z = (v, \theta)\). From the definition of \([z]_1\), (43) and (14) we have

\[
\frac{\zeta \lambda_1}{2} ||z||^2 = \frac{\zeta \lambda_1}{2} (|v|^2 + \gamma |\theta|^2) \leq \frac{\zeta}{2} (||v||^2 + \gamma ||\theta||^2) = \frac{\zeta}{2} ||z||^2.
\]

Using this and (39) we obtain

\[
[z]^2 \geq \zeta ||z||^2 - \frac{\zeta \lambda_1}{2} ||z||^2 \geq \frac{\zeta}{2} ||z||^2.
\]

Putting together (44) and (45) we obtain

\[
\frac{\zeta}{2} ||z||^2 \leq [z]^2 \leq (\max(\nu, \kappa) + c_r) ||z||^2, \quad \forall z \in V.
\]

Thus \([, ]\) defines an inner product in \(V\) with norm \([, ] = [\cdot, \cdot]^{1/2}\) equivalent to \(||\cdot||\).

Now let \(z(t) = (v(t), \theta(t))\) be a solution given by Theorem 1. Since \(z = (v, \theta) \in L^2(0, T; V)\) and \(z' = (v', \theta') \in L^2(0, T; V')\), we have

\[
\frac{1}{2} \frac{d}{dt} ||v||^2 = \langle v', v \rangle_{V', V} \quad \text{and} \quad \frac{1}{2} \frac{d}{dt} ||\theta||^2 = \langle \theta', \theta \rangle_{V', V}.
\]

Using (15) we have

\[
\frac{1}{2} \frac{d}{dt} ||z||^2 = \frac{1}{2} \frac{d}{dt} (|v|^2 + \gamma |\theta|^2) = \langle v', v \rangle_{V', V} + \gamma \langle \theta', \theta \rangle_{V', V} = \langle z', z \rangle_{V' \times V'}. \quad \text{So from (38) and (21) we have}
\]

\[
\frac{1}{2} \frac{d}{dt} ||z||^2 + \langle (A + R)z, z \rangle = \langle \Psi, z \rangle.
\]

From the definition of the norm \([, ]\) given by (42) we deduce that

\[
\frac{d}{dt} ||z||^2 + \zeta \lambda_1 ||z||^2 + 2[z]^2 = 2 \langle \Psi, z \rangle,
\]

for all \(t > 0\) in the distribution sense on \(\mathbb{R}^+\).

From (47) and using the equivalence of norms given by (46) one can deduce the classical estimates

\[
||z(t)||^2 \leq ||z_0||^2 e^{-\lambda_1 \zeta t} + \frac{2}{\lambda_1 \zeta} ||\Psi||_{L^2}^2, \quad \forall t \geq 0
\]

and

\[
\frac{1}{t} \int_0^t ||z(s)||^2 ds \leq \frac{2}{\zeta} ||z_0||^2 + \frac{4}{\zeta^2} ||\Psi||_{L^2}^2, \quad \forall t > 0.
\]
Thanks to Theorem 1 we can define a continuous semigroup \( \{ S(t) \}_{t \geq 0} \) in \( H \) by
\[
S(t)z_0 = z(t), \quad t \geq 0,
\]
where \( z \) is the solution of (37) with \( z(0) = z_0 \in H \). It is easy to see that the map \( S(t) : H \to H \), for \( t \geq 0 \), is Lipschitz continuous on bounded subsets of \( H \). Moreover from (48) it follows that that the set
\[
B = \left\{ z \in H : |z| \leq r_0 = \frac{1}{\zeta} \sqrt{\frac{2}{\lambda_1}} \| \Psi \|_{V'} \right\}
\]
is absorbing in \( H \) for the semigroup, and \( \zeta \) is given by (43).

We need the following weak continuity of the semigroup \( \{ S(t) \}_{t \geq 0} \).

**Lemma 4** Let \( \{ z_n \} \) be a sequence in \( H \) converging weakly in \( H \) to an element \( z_0 \in H \). Then
\[
S(t)z_n \to S(t)z_0 \text{ weakly in } H \quad \forall t \geq 0
\]
and
\[
S(t)z_n \to S(t)z_0 \text{ weakly in } L^2(0,T; V), \quad \forall T > 0.
\]

**Proof:** The proof is done following the steps in Rosa [17].

Let \( z_n(t) = S(t)z_n \) and \( z(t) = S(t)z_0 \). From (48) and (49) we find that
\[
\{ z_n \} \text{ is bounded in } z \in L^\infty(\mathbb{R}^+; H) \cap L^2(0,T; V), \quad \forall T > 0.
\]
Now from (38)
\[
z_n' = \Psi - (A + R)z_n - B(z_n),
\]
and since \( A \) and \( R \) are bounded linear operators from \( V \) into \( V' \), satisfying (16) and (34) respectively, and \( B \) satisfies (22), it follows that
\[
\{ z_n' \} \text{ is bounded in } L^2(0,T; V') \quad \forall T > 0.
\]

Then, for all \( w \in V \) and \( 0 \leq t \leq t + h \leq T \), with \( T > 0 \),
\[
(z_n(t+h) - z_n(t), w) = \int_t^{t+h} (z_n'(s), w) \, ds
\]
\[
\leq \|w\|_{h^{1/2}} \|z_n'\|_{L^2(0,T; V')}
\]
\[
\leq c_T \|w\|_{h^{1/2}},
\]
where \( c_T \) is positive, independent of \( n \). Then, for \( w = z_n(t+h) - z_n(t) \), which belongs to \( V \) for almost every \( t \), we find from (54) that
\[
|z_n(t+h) - z_n(t)|^2 \leq c_T h^{1/2} |z_n(t+h) - z_n(t)|.
\]
Hence
\[
\int_0^{T-h} |z_n(t+h) - z_n(t)|^2 \, dt \leq c_T h^{1/2} \int_0^{T-h} \|z_n(t+h) - z_n(t)\| \, dt.
\]
Using Cauchy-Schwarz inequality and (53) we find from (55) that
\[
\int_0^{T-h} |z_n(t+h) - z_n(t)|^2 \, dt \leq \bar{c}_T h^{1/2},
\]
for another positive constant \( \bar{c}_T \) independent of \( n \).

Now let us introduce the notation \( \Omega_r = \Omega \cap \{ x \in \mathbb{R}^2 : |x| < r \} \) and
\[
X_r = L^2(\Omega_r) \times L^2(\Omega_r), \quad Y_r = H_0^1(\Omega_r) \times H_0^1(\Omega_r).
\]
Introducing this notation, taking (56) in account, and applying Lebesgue’s Dominated Convergence Theorem, we obtain

$$
\lim_{h \to 0} \sup_n \int_0^{T-h} \|z_n(t+h) - z_n(t)\|_{L^p_X}^2 \, dt = 0,
$$

(57)

for all $r > 0$. We can find a smooth truncation function $\tau$ such that $w_{n,r}(x) = \tau(|x|/r)z_n(x)$ satisfies

$$
w_{n,r}(x) = \begin{cases} 
z_n(x); & \text{for } |x| \leq r/2 \\
0; & \text{for } |x| \geq r.
\end{cases}
$$

(58)

We fix $r_0 > 0$ and consider $r \geq r_0$, for which $\tau(|x|/r)$ and its derivatives are uniformly bounded in $x \in \mathbb{R}^2$ and in $r \geq r_0$. Then, from (57), we find that

$$
\lim_{h \to 0} \sup_n \int_0^{T-h} \|w_{n,r}(t+h) - w_{n,r}(t)\|_{L^2_X}^2 \, dt = 0,
$$

for all $T > 0$ and all $r > 0$, while from (53) we find that

$$
\{w_{n,r}\}_n \text{ is bounded in } L^\infty(\mathbb{R}^+; X_r) \cap L^2(0, T; Y_r)
$$

uniformly for all $T > 0$ and all $r \geq r_0$.

Thus, by a compactness theorem (see [15] Theorem 16.3 with $X = X_r, Y = Y_r$ and $p = 2$),

$$
\{w_{n,r}\}_n \text{ is relatively compact in } L^2(0, T; X_r) \quad \forall T > 0, \quad \forall r \geq r_0.
$$

(59)

It follows from (59) and (58) that

$$
\{z_{n, t}\}_n \text{ is relatively compact in } L^2(0, T; X_r) \quad \forall T > 0, \quad \forall r \geq r_0.
$$

(60)

Then from (53) and (60), and by a diagonal process, we can extract a subsequence $\{z_{n'}\}_{n'}$ such that

$$
z_{n'} \to \tilde{z} \quad \text{weak-star in } L^\infty(\mathbb{R}^+; H),
$$

$$
\text{weakly in } L^2_{\text{loc}}(\mathbb{R}^+; V),
$$

$$
\text{strongly in } L^2_{\text{loc}}(\mathbb{R}^+; X_r), \quad \forall r \geq r_0,
$$

(61)

for some

$$
\tilde{z} \in L^\infty(\mathbb{R}^+; H) \cap L^2_{\text{loc}}(\mathbb{R}^+; V).
$$

The convergence (61) allow us to pass to the limit in the equation for $z_{n'}$ to find that $\tilde{z}$ is a solution of (37) with $\tilde{z}(0) = z_0$. By the uniqueness of the solutions we must have $\tilde{z} = z$. Then by a contradiction argument we deduce that the whole sequence $\{z_n\}_n$ converges to some $z$ in the sense of (61). This proves (52).

Now, from the strong convergence in (61) we also have that $z_{n}(t)$ converges strongly in $X_r$ to $z(t)$ for almost every $t \geq 0$ and all $r \geq r_0$. Hence for all $w \in V \times D(\Omega)$,

$$
(z_n(t), w) \to (z(t), w), \quad \text{for a.e. } t \in \mathbb{R}^+.
$$

Moreover, from (53) and (54), we see that $\{(z_n(t), w)\}_n$ is equibounded and equicontinuous on $[0, T]$, for all $T > 0$. Therefore

$$
(z_n(t), w) \to (z(t), w), \quad \forall t \in \mathbb{R}^+, \forall w \in V \times D(\Omega).
$$

(62)

Finally, (51) follows from (62) by taking into account (53) and the fact that by definition $V \times D(\Omega)$ is dense in $V$.

$\square$
3 Global Attractor

In this Section we prove the existence of the global attractor.

**Definition 1** We say that a semigroup \( \{S(t)\}_{t \geq 0} \) is asymptotically compact in \( X \) if

\[
\{S(t_n)z_n\}_n \text{ is precompact in } X \text{ whenever }
\]

\[
\{z_n\}_n \text{ is bounded in } X \text{ and } t_n \to \infty.
\]

So let \( B \subset H \) be bounded and consider \( \{z_n\}_n \subset B \) and \( \{t_n\}_n, t_n \geq 0, t_n \to \infty \). Since the set \( B \) defined in (50) is absorbing, there exists a time \( T(B) > 0 \) such that

\[
S(t)B \subset B, \quad \text{for all } t \geq T(B).
\]

Hence for \( t_n \) large enough \( (t_n \geq T(B)) \),

\[
S(t_n)z_n \in B.
\]

Thus \( \{S(t_n)z_n\}_n \) is weakly precompact in \( H \) and

\[
S(t_{n'})z_{n'} \rightharpoonup w \text{ weakly in } H,
\]

for some subsequence \( n' \) and \( w \in B \) (since \( B \) is closed and convex).

From this weak convergence we have immediately that

\[
|w| \leq \liminf_{n'} |S(t_{n'})z_{n'}|.
\]

**Lemma 5** The sequence \( S(t_{n'})z_{n'} \) satisfies

\[
\limsup_{n'} |S(t_{n'})z_{n'}| \leq |w|.
\]

**Proof:** Using the variation of constants formula in the energy equation (47), and letting \( z(T) = S(T)\bar{z} \) we have

\[
|S(T)\bar{z}|^2 = e^{-\zeta \lambda t}|\bar{z}|^2 + 2 \int_0^T e^{-\zeta \lambda (T-s)} (\langle \Psi, S(s)\bar{z} \rangle - |S(s)\bar{z}|^2) \, ds
\]

for all \( \bar{z} \in H \), and \( T \geq 0 \).

For \( T \in \mathbb{N} \) and \( t_n > T \) we have by (55) letting \( \bar{z} = S(t_n - T)z_n \),

\[
|S(t_n)z_n|^2 = |S(T)S(t_n - T)z_n|^2 = e^{-\zeta \lambda t}|S(t_n - T)z_n|^2
\]

\[
+ 2 \int_0^T e^{-\zeta \lambda (T-s)} \langle \Psi, S(s)S(t_n - T)z_n \rangle \, ds
\]

\[
- 2 \int_0^T e^{-\zeta \lambda (T-s)} |S(s)S(t_n - T)z_n|^2 \, ds.
\]

We estimate now each one of the three terms above.

In the same way we have obtained (63), we also have, for each \( T > 0 \),

\[
S(t_n - T)z_n \in B,
\]

for \( t_n \geq T + T(B) \). Thus \( \{S(t_n - T)z_n\}_n \) is weakly precompact in \( H \) and by using a diagonal process and passing to a further subsequence if necessary we can assume that

\[
S(t_{n'})z_{n'} \rightharpoonup w_T \text{ weakly in } H, \quad \forall T \in \mathbb{N},
\]

with \( w_T \in B \).
Note then by the weak continuity of $S(t)$ established in Lemma 4 that
\[
w = \lim_{n'} H_\omega S(t_{n'}) z_{n'} = \lim_{n'} H_\omega S(T) S(t_{n'} - T) z_{n'} = S(T) \lim_{n'} H_\omega S(t_{n'} - T) z_{n'} = S(T) w_T,
\]
where $\lim H_\omega$ denotes the limit taken in the weak topology of $H$. Thus
\[
w = S(T) w_T, \quad \forall T \in \mathbb{N}. \quad (69)
\]
Hence from (67) we find
\[
\limsup_{n'} e^{-\zeta \lambda_1 (T-s)} |S(t_{n'} - T) z_{n'}|^2 \leq r_0^2 e^{-\zeta \lambda_1 T}. \quad (70)
\]
This takes care of the first term in (66). Let us estimate now the second term. From the weak continuity of Lemma 4 and (68)
\[
S(\cdot) S(t_{n'} - T) z_{n'} \rightharpoonup S(\cdot) w_T \text{ weakly in } L^2(0, T; V).
\]
Then since
\[
s \mapsto e^{-\zeta \lambda_1 (T-s)} \Psi \in L^2(0, T; V'),
\]
and (71) we find
\[
\lim_{n'} \int_0^T e^{-\zeta \lambda_1 (T-s)} \langle \Psi, S(s) S(t_{n'} - T) z_{n'} \rangle ds
\]
\[
= \int_0^T e^{-\zeta \lambda_1 (T-s)} \langle \Psi, S(s) w_T \rangle ds. \quad (72)
\]
This takes care of the second term in (66). Let us estimate now the last term. From (46) the norm $|\cdot|$ is equivalent to $\|\cdot\|$ in $V$. Also
\[
0 < e^{-\zeta \lambda_1 T} \leq e^{-\zeta \lambda_1 (T-s)} \leq 1, \quad \forall s \in [0, T],
\]
and therefore
\[
\left( \int_0^T e^{-\zeta \lambda_1 (T-s)} [\|s\|_V^2] ds \right)^{1/2}
\]
is a norm in $L^2(0, T; V)$ equivalent to the usual norm. Hence from (71) we deduce that
\[
\int_0^T e^{-\zeta \lambda_1 (T-s)} [S(s) w_T]^2 ds
\]
\[
\leq \liminf_n \int_0^T e^{-\zeta \lambda_1 (T-s)} [S(s) S(t_n - T) z_n]^2 ds.
\]
Therefore
\[
\limsup_{n'} \left( -2 \int_0^T e^{-\zeta \lambda_1 (T-s)} [S(s) S(t_{n'} - T) z_{n'}]^2 ds \right)
\]
\[
= -\liminf_{n'} 2 \int_0^T e^{-\zeta \lambda_1 (T-s)} [S(s) S(t_{n'} - T) z_{n'}]^2 ds
\]
\[
\leq -2 \int_0^T e^{-\zeta \lambda_1 (T-s)} [S(s) w_T]^2 ds. \quad (73)
\]
This takes care of the last term in (66).
4 DIMENSION OF GLOBAL ATTRACTOR

We can now take the limsup in (66), taking (70), (72) and (73) into account to obtain
\[
\limsup_{n'} |S(t_{n'}) z_{n'}|^2 \\
\leq r_0^2 e^{-c \lambda_1 T} + 2 \int_0^T e^{-c \lambda_1 (T-s)} \left( \langle \Psi, S(s) w_T \rangle - [S(s) w_T]^2 \right) \, ds.
\]
(74)

On the other hand, we obtain from (65) applied to \( w = S(T) w_T \) that
\[
|w|^2 = |S(T) w_T|^2 \\
\leq e^{-c \lambda_1 T} |w_T|^2 + 2 \int_0^T e^{-c \lambda_1 (T-s)} \left( \langle \Psi, S(s) w_T \rangle - [S(s) w_T]^2 \right) \, ds.
\]
(75)

From (74) and (75) we then find
\[
\limsup_{n'} |S(t_{n'}) z_{n'}|^2 \leq |w|^2 + (r_0^2 - |w_T|^2) e^{-c \lambda_1 T} \\
\leq |w|^2 + r_0^2 e^{-c \lambda_1 T}, \quad \forall T \in \mathbb{N}.
\]
(76)

Let \( T \) go to infinity in (76) to obtain
\[
\limsup_{n'} |S(t_{n'}) z_{n'}|^2 \leq |w|^2
\]
as claimed.

\[ \square \]

Lemma 5 and (64) implies that \( S(t_{n'}) z_{n'} \) converges in norm to \( w \). Since \( H \) is a Hilbert space and we have weak convergence from (63) we have that
\[ S(t_{n'}) z_{n'} \to w \text{ strongly in } H. \]

This shows that \( \{ S(t) z_n \}_n \) is precompact in \( H \) and hence that \( \{ S(t) \}_{t \geq 0} \) is asymptotically compact in \( H \). We state and prove now our main result:

**Theorem 2** Under the conditions of Theorem 1 the dynamical system \( \{ S(t) \}_{t \geq 0} \) associated with the evolution equation (37) possesses a global attractor in \( H \), i.e. a compact invariant set \( A \) in \( H \) which attracts all bounded sets in \( H \). Moreover, \( A \) is connected in \( H \) and is maximal for the inclusion relation among all the functional invariant sets bounded in \( H \).

**Proof:** We know that \( \{ S(t) \}_{t \geq 0} \) is a continuous asymptotically compact semigroup and has a bounded and connected absorbing set \( B \) in \( H \), a complete metric space. We know also that the map \( t \mapsto S(t) z_0 \) is continuous from \( \mathbb{R}^+ \) into \( H \) from Theorem 1. The proof follows from Temam [20, Theorem 1.1.1], with the attractor \( A = \omega(B) \), the omega limit of \( B \).

\[ \square \]

4 Dimension of Global Attractor

Following Temam [20], we estimate the dimension of the global attractor \( A \) for the system under consideration.

Let \( z_0 \in H \) and set \( z(t) = S(t) z_0 \), for \( t \geq 0 \). From (38) we see that the linearized flow around \( z \) is given by the equation
\[
\begin{cases}
Z' + (A + R)Z + B(z, Z) + B(Z, z) = 0 & \text{in } V', \\
Z(0) = \xi.
\end{cases}
\]
(77)

By the same technique used for the nonlinear problem, one can show the following lemma:

**Lemma 6** Given \( \xi \in H \), there exists a unique \( Z \in L^\infty(0, T; H) \cap L^2(0, T; V) \), for all \( T > 0 \), satisfying (77). Moreover, \( Z' \in L^2(0, T; V') \) and \( Z \in C([0, T]; H) \) for all \( T > 0 \).
We can then define a linear map \( L(t; z_0) : H \to H \) by setting \( L(t; z_0)\xi = Z(t) \). It can also be proven that \( L(t; z_0) \) is bounded and that \( \{S(t)\}_{t \geq 0} \) is uniformly differentiable on \( A \), i.e.

\[
\lim_{\varepsilon \to 0} \sup_{0 < |w - z| \leq \varepsilon} \frac{|S(t)w_0 - S(t)z_0 - L(t; z_0) \cdot (w_0 - z_0)|}{|w_0 - z_0|} = 0.
\]

Write (77) as

\[
Z' = F'(z)Z \equiv -(A + R)Z - B(z, Z) - B(Z, z),
\]

and define numbers \( q_m \), for \( m \in \mathbb{N} \), by

\[
q_m = \lim_{t \to \infty} \sup_{z_0 \in A} \sup_{\xi_1, \ldots, \xi_m} \frac{1}{t} \int_0^t \text{Tr}(F'(S(\tau)z_0) \circ Q_m(\tau)) \, d\tau,
\]

where \( Q_m(\tau) = Q_m(\tau; z_0, \xi_1, \ldots, \xi_m) \) is the orthogonal projector in \( H \) onto the space spanned by

\( L(t; z_0)\xi_1, \ldots, L(t; z_0)\xi_m \).

The trace (denoted by \( \text{Tr} \)) of \( F'(S(t)z_0) \circ Q_m(t) \) in (79) is defined at least almost everywhere in \( t \).

From the general result in Temam [20, Section V.3.4], we have that if \( q_m < 0 \) for some \( m \in \mathbb{N} \) then the global attractor has finite Hausdorff and fractal dimensions estimated, respectively, as

\[
\dim_H(A) \leq m,
\]

\[
\dim_F(A) \leq m \left( 1 + \max_{1 \leq j \leq m} \frac{(q_j)_+}{|q|} \right).
\]

We want to relate the dimension of the global attractor with the physical parameters. Note that \( \lambda_{-1} \) is related to the geometry of the domain and it has the dimension of area as one can see from (14). Its square root is a characteristic length for the problem.

Let us introduce the generalized Grashof (Gr), Reynolds (Re) and Rayleigh (Ra) (non-dimensional) numbers, defined by

\[
\text{Gr}_u = \frac{\|f_u\|_{L^2}}{\nu \lambda_1^2}, \quad \text{Ra}' = \frac{|\sigma|^{\frac{3}{2}} |f_T|^{\frac{3}{2}}}{\nu \kappa \lambda_1},
\]

\[
\text{Re}' = \frac{\|\Delta u_b\|_{L^2}}{\nu \lambda_1^2}, \quad \text{Ra}'' = \frac{|\sigma| |\Delta T_b|_{L^2}}{\nu \kappa \lambda_1},
\]

\[
\text{Re}'' = \frac{\epsilon b_{in}}{\nu \lambda_1^2}, \quad \text{Ra}''' = \frac{|\sigma| |T_b - T_r|_{L^2}}{\nu \kappa \lambda_1},
\]

\[
\text{Re} = \max(\text{Re}', \text{Re}'') \quad \text{Ra} = \max(\text{Ra}', \text{Ra}'', \text{Ra}''', \text{Ra}'''),
\]

and the (non-dimensional) Prandtl number (Pr) defined as

\[
\text{Pr} = \frac{\nu}{\kappa}.
\]

**Theorem 3** Under the conditions of Theorem 1, there exists an absolute constant \( C \) such that the Hausdorff dimension of the universal attractor for the Bénard problem is bounded by

\[
\dim_H(A) \leq 1 + C(1 + \text{Pr}^{1/2}) \{ \text{Pr}(\text{Gr}_u^2 + \text{Re}'^2 + \text{Re}''^2) + \text{Pr}^{-1} (\text{Ra}^2 + \text{Ra}''^2) + \text{Ra}^3 + \text{Ra}''^3 \}
\]

\[
\leq 1 + C(1 + \text{Pr}^{1/2}) \{ \text{Pr}(\text{Gr}_u^2 + \text{Re}'^4) + \text{Pr}^{-1} \text{Ra}^2 + \text{Ra}^3 \}.
\]
If the domain has finite measure $|\Omega| < \infty$, the Hausdorff dimension of the universal attractor is bounded by
\[
\dim_H(A) \leq 1 + C(1 + Pr^{1/4})\{Pr^{1/2}(Gr_u + Re') + Pr^{-1/2}(Ra'' + Ra''') + Ra^{3/2} + Ra^{3/2}/2\}
\leq 1 + C(1 + Pr^{1/4})\{Pr^{1/2}(Gr_u + Re') + Pr^{-1/2}Ra + Ra^{3/2}\}
\]
with $|\Omega|$ replacing $1/\lambda_1$ in the definition above.

**Remark:** The Reynolds and Raleigh numbers (except $Ra'$) defined above are based on the background flow and temperature. In the case in which the background flow and temperature can be constructed from given conditions on the boundary, it is physically more relevant to estimate the dimension in terms of new Reynolds and Raleigh numbers based on the boundary data. In this case, the dimension estimate above may be of different order, even exponential (see Section 5).

**Remark:** We can obtain similar estimates for the fractal dimension of the universal attractor as well.

**Proof:** In order to estimate the numbers $q_m$, let $z_0 \in A$ and $\xi_1, \ldots, \xi_m \in H$. Set $z(t) = S(t)z_0$ and $Z_j(t) = L(t; z_0)\xi_j, t \geq 0$. Let $\{w_j(t), \varphi_j(t)\}_{j=1}^{m}$, be a basis for $\text{Span}\{Z_1(t), \ldots, Z_m(t)\}$ such that $\{w_j\}_{j=1}^{m}$ is orthonormal in $H_1$ and $\{\varphi_j\}_{j=1}^{m}$ is orthonormal in $H_2$. Let us define $\varphi_j = (w_j, \psi_j) = (w_j/\sqrt{\tau}, \psi_j/\sqrt{\tau})$. An easy computation yields $\{\varphi_j\}_{j=1}^{m}$ is orthonormal in $H$. Since $Z(t) \in V$ (at least almost everywhere in $t$) we can assume that $\varphi_j(t) \in V$ (by the Gram-Schmidt orthogonalization process). Then we have
\[
\text{Tr}(F'(z(t)) \circ Q_m(t)) = \sum_{j=1}^{m} \langle F'(z(t)) \varphi_j, \varphi_j \rangle_{V, V}
= \langle (z(x) - z(t)) - B(x, \varphi_j, z) \rangle_{V, V}
\leq \langle \nu \sum_{j=1}^{m} |w_j| \rangle_{V, V} \leq \langle \nu \sum_{j=1}^{m} |w_j| \rangle_{V, V}
\]

Now let $\rho(x) = \sum_{j=1}^{m} \langle \nu^{1/2} |w_j| \rangle_{V, V} + \gamma \nu^{1/2} \langle |\psi_j| \rangle_{V, V}$. A standard computation (see [17] e.g.) and the definition of $\rho$ yields
\[
\sum_{j=1}^{m} b_1(w_j, v, w_j) \leq \frac{\|v\|}{\nu^{1/2}} \left| \sum_{j=1}^{m} |w_j| \right|_{L^2} \leq \frac{\|v\|}{\nu^{1/2}} \|\rho\|_{L^2}.
\]

Applying Cauchy-Schwarz and Young’s inequality point-wise, we obtain
\[
|w_j(x) \cdot \nabla \theta(x)| |\psi_j(x)| \leq \gamma |\nabla \theta(x)| |w_j(x)| |\psi_j(x)| \leq \frac{\gamma^{1/2}}{2(\nu \kappa) \sqrt{4 \nu \kappa^{1/2}}} |\nabla \theta(x)| (|\nabla |w_j(x)||^2 + \gamma \nu^{1/2} |\psi_j(x)|^2).
\]

Integrating this expression in $x$, summing it in $j$ from 1 up to $m$ we obtain
\[
\gamma \sum_{j=1}^{m} b_2(w_j, \theta, \psi_j)
= \gamma \sum_{j=1}^{m} \int_{\Omega} w_j(x) \cdot \nabla \theta(x) \psi_j(x) \, dx \leq \frac{\gamma^{1/2}}{2(\nu \kappa) \sqrt{4 \nu \kappa^{1/2}}} \int_{\Omega} |\nabla \theta(x)| \left| \sum_{j=1}^{m} (\nu^{1/2} |w_j(x)|^2 + \gamma \nu^{1/2} |\psi_j(x)|^2) \right| \, dx
\]
\[\begin{align*}
\gamma \frac{m}{\sum_{j=1}^{m} b_2(w_j, \theta, \psi_j)} & \leq \frac{\gamma^{1/2}}{2(\kappa \nu)^{1/2}} |\theta| |\rho| L^2. \\
\end{align*}\]

Therefore we have

\[\left| \sum_{j=1}^{m} b_2(w_j, \theta, \psi_j) \right| \leq \frac{\gamma^{1/2}}{2(\kappa \nu)^{1/2}} |\theta| |\rho| L^2. \quad (85)\]

Hence from (84) and (85)

\[\left| \sum_{j=1}^{m} b(\varphi_j, z, \varphi_j) \right| \leq |\rho| L^2 \left( \frac{||v||}{\nu^{1/2}} + \frac{\gamma^{1/2}}{2(\kappa \nu)^{1/2}} |\theta| \right) \quad (86)\]

From the definition of \( \rho, \bar{w}_j \) and \( \bar{\psi}_j \) we observe that

\[\rho(x) = \frac{1}{2} \sum_{j=1}^{m} (\nu^{1/2}|\bar{w}_j(x)|^2 + \kappa^{1/2}|\bar{\psi}_j(x)|^2).\]

The generalized Lieb-Thirring inequality (see [7]) can be applied to the orthonormal finite families \( \{\bar{w}_j\}_j \) and \( \{\bar{\psi}_j\}_j \). This guarantees the existence of a constant \( \tilde{k} \) independent of the number of functions \( m \) such that

\[|\rho|_{L^2} \leq \frac{1}{2} \left( \frac{\nu}{\nu^{1/2}} \left( \sum_{j=1}^{m} |\bar{w}_j|^2 \right)^2 + \kappa \left( \sum_{j=1}^{m} |\bar{\psi}_j|^2 \right)^2 \right) \leq \tilde{k} \sum_{j=1}^{m} (\nu||w_j||^2 + \kappa||\psi_j||^2) \]

\[= \tilde{k} \sum_{j=1}^{m} (\nu||w_j||^2 + \gamma\kappa||\psi_j||^2) \quad (87)\]

Inserting (87) into (86) and using Young’s inequality we obtain

\[\left| \sum_{j=1}^{m} b(\varphi_j, z, \varphi_j) \right| \leq \frac{\tilde{k}}{2} \left( \frac{||v||^2}{\nu} + \frac{\gamma}{4(\kappa \nu)^{1/2}} |\theta|^2 \right) + \frac{\delta}{2} \sum_{j=1}^{m} (\nu||w_j||^2 + \gamma\kappa||\psi_j||^2) \]

Using this inequality in (82) we obtain

\[\text{Tr}(F^*(z(\tau)) \circ Q_m(\tau)) \leq \frac{\tilde{k}}{2\delta} \left( \frac{||v||^2}{\nu} + \frac{\gamma}{4(\kappa \nu)^{1/2}} |\theta|^2 \right) - \frac{\delta}{2} \sum_{j=1}^{m} (\nu||w_j||^2 + \gamma\kappa||\psi_j||^2) \quad (89)\]

Since \( \{\varphi_j\}_j=1 \ldots m \) is orthonormal in \( H \),

\[|w_j|^2 = \gamma|\psi_j|^2 = 1/2.\]
Using this and the Poincaré inequality (14) in (89) we obtain
\[
\text{Tr}(F'(z(\tau)) \circ Q_m(\tau)) \leq \frac{k}{2\delta} \left( \frac{\|v\|^2}{\nu} + \frac{\gamma}{4(\nu \kappa)^{1/2}} \|\theta\|^2 \right) - m \frac{\delta \lambda_1}{4}(\nu + \kappa).
\]

Using this inequality in (79) we find that
\[
q_m \leq \frac{k}{2\delta} \limsup_{t \to \infty} \sup_{z_0 \in A} \frac{1}{t} \int_0^t \left( \frac{\|v\|^2}{\nu} + \frac{\gamma}{4(\nu \kappa)^{1/2}} \|\theta\|^2 \right) \, dt
- m \frac{\delta \lambda_1}{4}(\nu + \kappa), \quad \forall m \in \mathbb{N}.
\]

Let us now estimate the last term of the inequality above. We could have used (49), but we want to be more precise.

From (6) and (7), using Lemma 2 (orthogonality of \(b\)) and estimates (31) and (32), we obtain the following energy estimates
\[
\frac{1}{2} \frac{d}{dt} |v|^2 + \frac{3\nu}{16} \|v\|^2 \leq \frac{8}{\nu} \|\tilde{f}_u\|^2_{\nu} + \left( \frac{\sigma}{\lambda_1} \right)^2 \frac{1}{2\nu} \|\theta\|^2,
\]
\[
\frac{1}{2} \frac{d}{dt} |\theta|^2 + \frac{7\kappa}{8} \|\theta\|^2 \leq \frac{1}{\kappa} \|\tilde{f}_T\|^2_{\nu} + \frac{\beta^2}{8\kappa} \|v\|^2.
\]

Multiplying (92) by \(\gamma\), adding up with (91) and using (13) and (33) we obtain
\[
\frac{1}{2} \frac{d}{dt} |z|^2 + \frac{\nu}{16} \|v\|^2 + \frac{3\gamma \kappa}{8} \|\theta\|^2 \leq \frac{8}{\nu} \|\tilde{f}_u\|^2_{\nu} + \frac{1}{\kappa} \|\tilde{f}_T\|^2_{\nu}.
\]

From this it follows that there exists an absolute constant \(C'\), independent of the physical parameters, such that
\[
\limsup_{t \to \infty} \sup_{z_0 \in A} \frac{1}{t} \int_0^t \|v\|^2 \, dt \leq C' \left( \frac{\|\tilde{f}_u\|^2_{\nu}}{\nu} + \frac{\gamma \|\tilde{f}_T\|^2_{\nu}}{\nu \kappa} \right),
\]
\[
\limsup_{t \to \infty} \sup_{z_0 \in A} \frac{\gamma}{t} \int_0^t \|\theta\|^2 \, dt \leq C' \left( \frac{\|\tilde{f}_u\|^2_{\nu}}{\nu \kappa} + \frac{\gamma \|\tilde{f}_T\|^2_{\nu}}{\kappa} \right).
\]

Since \(\gamma\) satisfies (13) and we want to minimize the last term, we set henceforth \(\gamma = (\sigma/\lambda_1)^2/(\nu \kappa)\). Therefore we have
\[
\limsup_{t \to \infty} \sup_{z_0 \in A} \frac{1}{t} \int_0^t \left( \frac{\|v\|^2}{\nu} + \frac{\gamma}{4(\nu \kappa)^{1/2}} \|\theta\|^2 \right) \, dt
\leq C' \left( 1 + \Pr^{3/2} \right) \left( \frac{\|\tilde{f}_u\|^2_{\nu}}{\nu^3} + \frac{\sigma^2 \|\tilde{f}_T\|^2_{\nu}}{\lambda_1^2 \nu^3 \kappa^2} \right).
\]

Let \(m' \in \mathbb{N}\) be defined by
\[
m' - 1 \leq C' \left( 1 + \Pr^{3/2} \right) \left( \frac{\|\tilde{f}_u\|^2_{\nu}}{\nu^3} + \frac{\sigma^2 \|\tilde{f}_T\|^2_{\nu}}{\lambda_1^2 \nu^3 \kappa^2} \right) < m'.
\]

Applying (93) into (90), we deduce that \(q_{m'} < 0\). Therefore, from (80), and since
\[
\frac{1 + \Pr^{3/2}}{\nu + \kappa} \leq \frac{1}{\kappa} (1 + \Pr^{1/2}),
\]
\[
\dim_H(A) \leq m' \leq 1 + C' \left( 1 + \Pr^{1/2} \right) \left( \frac{\|\tilde{f}_u\|^2_{\nu}}{\nu^3} + \frac{\sigma^2 \|\tilde{f}_T\|^2_{\nu}}{\lambda_1^2 \nu^3 \kappa^2} \right).
\]
From (9), (10) and (29) we have that
\[
\begin{align*}
\|\tilde{f}_u\|_{V_j^2} & \leq \|f_u\|_{V_j^2} + \nu \|\Delta u_6\|_{V_j^2} + c_{\phi_{\nu}} + \sigma \|T_0 - T_r\|_{V_j^2}, \\
\|\tilde{f}_r\|_{V_j^2} & \leq \|f_r\|_{V_j^2} + \kappa \|\Delta T_0\|_{V_j^2} + c_{\phi_{\kappa}}.
\end{align*}
\] (95) (96)

Using these inequalities in (94) we obtain
\[
\dim_H(\mathcal{A}) \leq 1 + C' \left( \frac{1 + \Pr^{1/2}}{\lambda_1 \kappa} \right) \left\{ \frac{\sigma^2}{\lambda_1 \nu^3 \kappa^2} \left( \|\tilde{f}_r\|_{V_j^2}^2 + \kappa^2 \|\Delta T_0\|_{V_j^2}^2 + c_{\phi_{\kappa}}^2 \right) + \frac{1}{\nu^2} \left( \|f_u\|_{V_j^2}^2 + \nu^2 \|\Delta u_6\|_{V_j^2}^2 + c_{\phi_{\nu}}^2 + \sigma^2 \|T_0 - T_r\|_{V_j^2}^2 \right) \right\}.
\]

The first part of this theorem follows using the definition of the Grashof, Reynolds, Rayleigh and Prandtl numbers.

In case the domain has finite measure $|\Omega| < \infty$, we can improve the Hausdorff dimension estimate. This improvement is achieved with a more favorable estimate for $\sum_{j=1}^{m} \nu \|w_j\|^2 + \gamma \|\psi_j\|^2$, thanks to the asymptotic distribution of the eigenvalues for the 2D Laplace's and Stokes' operators for domains with finite measures (see [9]). Since $\bar{w}_j$ and $\psi_j$ are orthonormal, one can show as in [9] that there exists a constant $C''$, independent of the physical constants, such that
\[
\sum_{j=1}^{m} \|\bar{w}_j\|^2 \text{ and } \sum_{j=1}^{m} \|\psi_j\|^2 \geq \frac{2C'' m^2}{|\Omega|}.
\]

From this, the definition of $w_j$ and $\psi_j$ we have
\[
\sum_{j=1}^{m} \nu \|w_j\|^2 + \gamma \|\psi_j\|^2 = \sum_{j=1}^{m} \left( \frac{\nu \|\bar{w}_j\|^2}{2} + \gamma \frac{\|\psi_j\|^2}{2 \gamma} \right) \geq \frac{C'' m^2}{|\Omega|} (\nu + \kappa).
\]

Using this in (89) we obtain
\[
\text{Tr}(F'(\tau(\tau)) \circ Q_m(\tau)) \leq \frac{k}{2\delta} \left( \frac{\|\nu\|^2}{\nu} + \frac{\gamma}{4(\nu \kappa)^{1/2}} \|\theta\|^2 \right) - \frac{C'' m^2}{|\Omega|} (\nu + \kappa).
\]

Using this inequality and (93) in (79) we find that
\[
q_m \leq -\frac{C'' m^2}{|\Omega|} (\nu + \kappa) + C' (1 + \Pr^{3/2}) \left( \frac{\|\tilde{f}_u\|_{V_j^2}^2}{\nu^3} + \frac{\sigma^2 \|\tilde{f}_r\|_{V_j^2}^2}{\lambda_1 \nu^3 \kappa^2} \right), \forall m \in \mathbb{N}
\]

From [9], $1/\lambda_1 \leq 2\pi |\Omega|$. The rest of the proof is just like the end of first part of the proof.

\[
\square
\]

5 Applications

One way to satisfy (23) – (29) is to construct $z_6 = (u_6, T_6)$ such that
\[
\begin{align*}
T_6 & \in H^1(\Omega), \\
u_6 & \in L^\infty(\Omega) \cup L^4(\Omega), \\
\nabla u_6 & \in L^2(\Omega), \quad \nabla \cdot u_6 = 0.
\end{align*}
\] (97) (98) (99)

If $z_6 = (u_6, T_6)$ also satisfies (31) and (32) then, by Lemma 3, condition (39) holds. Section 5.1 is done in this context.

In Section 5.2 we study an example in which (23) – (29) are satisfied even though $\nabla u_6$ is not necessarily in $L^2(\Omega)$.

The first two Subsections, 5.1 and 5.2, show how to construct $z_6$ for bounded domains with some regularity. For certain nonsmooth and/or unbounded domains it is still possible to construct a suitable background flow and temperature which render $A + R$ V-elliptic. These are discussed in the final subsections.
5.1 Smooth and Bounded Ω with \( \phi_b \in H^{1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega) \) and \( \int_{\partial \Omega} \phi_u \cdot n = 0 \)

For \( \Omega \) smooth, \( \phi_0 = (\phi_u, \phi_T) \in H^{1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega) \) and \( \int_{\Gamma} \phi_u \cdot n d\Gamma = 0 \), where \( \Gamma \) is any connected component of \( \partial \Omega \), we can construct a suitable background flow and temperature from the following general result due to E. Hopf [8] (see also [19, pag. 173 and pag. 470]).

**Lemma 7 (Hopf)** Suppose \( \Omega \) is smooth and bounded. Let \( \Gamma_i \) be the connected components of \( \Gamma = \partial \Omega \). Given \( \phi_u \in H^{1/2}(\Gamma) \) such that \( \int_{\Gamma_i} \phi_u \cdot n d\Gamma = 0 \), and for any \( \delta > 0 \), there exists \( u_b \in H^1(\Omega) \) such that \( u_b = \phi_u \) on \( \Gamma \), \( \nabla \cdot u_b = 0 \), and

\[
\left| \int (v \cdot \nabla) u \cdot u_b \right| \leq \delta ||v||^2, \quad \forall v \in V_i.
\]

Following the proof of the lemma above we can construct a background temperature, as stated in the next lemma.

**Lemma 8** Suppose \( \Omega \) is smooth and bounded. Given \( \phi_T \in H^{1/2}(\partial \Omega) \), for any \( \delta > 0 \), there exists \( T_b \in H^1(\Omega) \) such that \( T_b = \phi_T \) on \( \partial \Omega \),

\[
\left| \int (v \cdot \nabla) \theta \cdot T_b \right| \leq \delta ||v|| ||\theta||, \quad \forall v \in V_i, \quad \forall \theta \in V_2.
\]

Putting together both lemmas, we obtain \( z_b \in H^1(\Omega) \times H^1(\Omega) \), so that (97)-(99) are satisfied. This implies that (23)-(29) hold. These lemmas also show that (31) and (32) hold provide we choose \( \delta \) sufficiently small. Therefore we can apply Theorems 2 and 3 in this case.

If we replace the Reynolds and Raleigh numbers based on the background flow and temperature by numbers based on the boundary data, current dimension estimates of the attractor yield an exponential dependence on these numbers (see [20]). The dimension estimate can be improved when the fluid flow is zero pointwise through the boundary, as we will see in Section 5.2.

5.2 Lipschitz and Bounded Ω with \( \phi_b \in L^\infty(\partial \Omega) \times L^\infty(\partial \Omega) \) and \( \phi_u \cdot n = 0 \) on \( \partial \Omega \)

This application is based on the work of Brown, Perry and Shen [3] which extended to the nonsmooth setting the work of Miranville and Wang [16]. Assume that \( \Omega \) is a simply connected Lipschitz (bounded) domain and that we are given \( \phi_b = (\phi_u, \phi_T) \in L^\infty(\partial \Omega) \times L^\infty(\partial \Omega) \) and \( \phi_u \cdot n = 0 \) a.e. on \( \partial \Omega \).

Note that this application is different from the one given in Section 5.1 since \( H^{1/2}(\partial \Omega) \) and \( L^\infty(\partial \Omega) \) are not related by set inclusions. It differs also, in a much more important way for applications, since we require that the fluid flow is zero pointwise through the boundary, in contrast with Section 5.1, where it is zero as an average only. For instance, it does not cover the case of a channel like in Section 5.3.

In this application, we improve the dimension estimate of Theorem 3 with respect to Reynolds and Raleigh numbers based on the boundary data, in contrast with Section 5.1, in which the estimate is exponential on these numbers.

Following [3] we can find \( z_b \) such that (23)-(29) are satisfied and \( z_b = \phi_b \) on \( \partial \Omega \) almost everywhere in the sense of nontangential convergence.

More precisely, in [3] they study the Navier-Stokes equation and construct the background velocity field \( u_b \in C^\infty(\Omega) \cap H^{1/2}(\Omega) \cap L^\infty(\Omega) \), such that (31) holds, \( \nabla \cdot u_b = 0 \) and

\[
||u_b||_{L^\infty(\Omega)} \leq C ||u_b||_{L^\infty(\partial \Omega)}
\]

for some constant \( C \). From this inequality we obtain

\[
\left| \int (u_b \cdot \nabla) \nabla \cdot v \right| + \left| \int (v \cdot \nabla) u_b \right| \leq 2C ||u_b||_{L^\infty(\partial \Omega)} ||v|| ||v||.
\]

They also show that

\[
\left| \int (u_b \cdot \nabla) \nabla \cdot v \right| \leq C ||u_b||_{L^\infty(\partial \Omega)} ||v||^{3/2}
\]

\[
(101)
\]

(102)
and one can check from their proof that
\[ \| \Delta u_b \|_{V'} \leq C \left\{ \frac{|\partial \Omega|^{1/2}}{\nu^{1/2}} \| u_b \|_{L^2(\Omega)}^{3/2} + \| u_b \|_{L^2(\Omega)} \right\}. \]  

(103)

Following the proof in [3] we can construct a background temperature \( T_b \in C^\infty(\Omega) \cap H^{1/2}(\Omega) \cap L^\infty(\Omega) \), such that (32) holds, and
\[ \| T_b \|_{L^\infty(\Omega)} \leq C \| T_b \|_{L^{-\infty}(\Omega)} \]
for some constant \( C \). From this inequality and (100) we obtain
\[ \left| \int (u_b \cdot \nabla) \tilde{\theta} \cdot T_b \right| \leq C \| \theta \|_{L^2(\Omega)} \| \tilde{\theta} \|_{L^2(\Omega)} \]
and
\[ \| \tilde{\theta} \|_{L^2(\Omega)} \]
\[ \| \Delta T_b \|_{V'} \leq C \left\{ \left( \frac{|\partial \Omega|}{\nu \lambda_1} \right)^{1/2} \| T_b \|_{L^2(\Omega)}^{3/2} + \| T_b \|_{L^2(\Omega)} \right\}. \]  

(105)

Now we use (104), (105), (102) and (103) to obtain (25), (26) and (29). Also, since \( T_b \in H^{1/2}(\Omega) \), condition (27) is satisfied. Therefore we can apply Theorems 2 and 3 here.

From [9], \( 1/\lambda_1 \leq 2\pi|\Omega| \). Defining
\[ \tilde{\text{Re}} = \frac{\| u_b \|_{L^2(\Omega)}}{\nu}, \quad \tilde{\text{Ra}} = \frac{|\sigma\| T_b \|_{L^2(\Omega)}}{\nu K}, \]
replacing \( c_{\text{nc}}, c_{b, \text{nt}}, \| \Delta u_b \|_{V'} \) and \( \| \Delta T_b \|_{V'} \) by the inequalities above, replacing \( 1/\lambda_1 \) by \( |\Omega| \) in the definition of the Grashof and Raleigh numbers, we obtain
\[ \text{dim}_H(A) \leq 1 + C(1 + \text{Pr}^{1/3}) \{ \text{Pr}^{1/2} (\text{Gr} + \tilde{\text{Re}}^{3/2} + \tilde{\text{Ra}}^{-1/2}) + \text{Pr}^{-1/2} (\tilde{\text{Ra}}^{3/2} + \text{Ra}^{3/2}) \}. \]

The constant \( C \) depends on the scale-invariant quantity \( \frac{|\partial \Omega|^2}{|\Omega|^2} \).

5.3 Channel with Irregularities and a Cavity

We apply our result for a channel with a cavity like the one shown in Figure 1.

With respect to the geometry of \( \Omega \) we assume that \( AB, AC \) and \( CD \) are straight line segments, the boundary is straight at the bottom near \( B \) and \( D \) and we have a right angle at each corner \( (A, B, C \) and \( D) \).

For the velocity given at the boundary, we assume that:
- a) it has a constant horizontal velocity \( u = U_c \) on \( AC \), the top of the channel,
- b) we have a no-slip boundary condition \( u = 0 \) at the bottom of the channel,
- c) it is equal to a Poiseuille flow \( u_{pf} \) (a second degree polynomial) matching the conditions given above, on \( AB \) and \( CD \), the input and output of the channel.

For the temperature given at the boundary, we assume that:
- a) it equals \( T_{up} \) (a constant) on \( AC \), the top of the channel,
b) it equals $T_{\text{down}}$ (a constant) at bottom of the channel,

c) it equals $T_i$, a linear interpolation of $T_{\text{up}}$ and $T_{\text{down}}$ on $AB$ and $CD$, the input and output of the channel.

Since $u_b \cdot n$ is not zero on $\partial \Omega$, we cannot apply the method in Section 5.2. Due to the irregularities, we cannot apply directly, the results of Section 5.1 either.

By adapting E. Hopf's construction for smooth domains (see [19] and [13]), we can find $u_b$ and $T_i$, smooth functions defined on $\Omega$, with $\nabla \cdot u_b = 0$, satisfying the boundary conditions above, and satisfying conditions (31) and (32).

We outline this construction for the velocity field only, since for the temperature it is similar. We define the function $\lambda \rightarrow \xi_\epsilon(\lambda)$, for $\lambda \geq 0$, by

$$\xi_\epsilon(\lambda) = \begin{cases} 1 & \text{if } \lambda < \delta(\epsilon)^2 \\ \varepsilon \log \left( \frac{\delta(\epsilon)}{\lambda} \right) & \text{if } \delta(\epsilon)^2 < \lambda < \delta(\epsilon) \\ 0 & \text{if } \lambda > \delta(\epsilon) \end{cases} \tag{106}$$

where $\delta(\epsilon) = \exp(-1/\epsilon)$. We consider a regularization $\theta_\epsilon$ of $\xi_\epsilon$ which is a $C^\infty$ function satisfying

$$|d\theta_\epsilon(\lambda)/d\lambda| \leq \varepsilon/\lambda, \quad \theta_\epsilon(\lambda) = 1, \quad \text{for } \lambda \leq \delta(\epsilon)^2/2,$$

and

$$\theta(\lambda) = 0, \quad \text{for } \lambda \geq 2\delta(\epsilon).$$

Let $\zeta$ be such that $u_p = \text{curl } \zeta$ (note that $\zeta$ only depends on $x_2$). By using a partition of unity subordinate to a covering of the closure $\Omega$ and local coordinates, we reduce the problem to five types of subdomains. We define $u_b$ locally on each sub-domain and extend it globally using this partition of unity. The types of subdomains are the following:

a) interior of the channel. Here we set $u_b = 0$.

b) bottom of the channel, away from $AB$ and $CD$. Here we set $u_b = 0$.

c) input/output of the channel, away from the corners. Assume we are dealing with $AB$ and $B$ is the origin. Then set $u_b(x) = \text{curl } (\theta_\epsilon(x_1)\zeta(x_2))$.

d) top of the channel, away from $AB$ and $CD$. Set $u_b = \theta_\epsilon(x_2)U\zeta_1$.

e) corner. Assume, without loss of generality, that we are dealing with corner $B$ and $B$ is the origin. Then set $u_b(x) = \text{curl } (\theta_\epsilon(x_1)x_2)\zeta(x_2))$.

Subdomains a) and b) present no difficulty since the boundary conditions and $u_b$ are both zero. Subdomains c) and d) are smooth and can be considered exactly as in Hopf's original construction. The only novelty/difficulty are in the corners. For this reason we present, only for the subdomains in case e), the details on how to obtain the background flow $u_b$, which has to be smooth, with divergence zero, and satisfying condition (31).

Proof: (part e) corner) We consider only corner $B$, the others are similar. We assume, thanks to the partition of unity that $\Omega = (0,\infty) \times (0,\infty)$, and that $\zeta = \zeta(x)$ has been extended to a compactly supported $C^\infty$ function on $\mathbb{R}^2$ whose curl $\zeta$ coincides with the Poiseuille flow in a neighborhood of $B$. We set

$$\psi(x) = u_b(x) = \text{curl } (\theta_\epsilon(x_1)x_2)\zeta(x)).$$

We show now that this choice of $u_b$ has the desired properties.
First, let us recall the classical Hardy inequality
\[ \int_0^{+\infty} \left( \frac{f(s)}{s} \right)^2 ds \leq 2 \int_0^{+\infty} |f'(s)|^2 ds, \quad \forall f \in \mathcal{D}(0, +\infty). \] (107)

From the definition of \( \theta_\varepsilon \), since it is a regularization of \( \chi_\varepsilon(x) = \xi_\varepsilon(x_1, x_2) \), we have that \( |D_i \theta_\varepsilon(x)| \leq \varepsilon/x_i \), \( \theta_\varepsilon = 1 \) for \( x_i = 0 \) and \( \theta_\varepsilon = 0 \) for \( x_1 x_2 > 2\delta(\varepsilon) \). Now let \( \psi = u_0 = \text{curl} (\theta_\varepsilon \zeta) \). It is easy to check that \( u_0 = u_p \) for \( x_i = 0 \). Therefore the boundary conditions are satisfied. Also
\[
|\psi_2(x)| = |(D_1 \theta_\varepsilon) \zeta + \theta_\varepsilon (D_1 \zeta)| \leq \frac{\varepsilon}{x_1} |\zeta| + |\theta_\varepsilon||D_1 \zeta| \\
\leq \text{(since } \theta_\varepsilon \leq 1 \text{ and } \zeta \in L^\infty) \\
\leq c \left( \frac{\varepsilon}{x_1} + |D_1 \zeta| \right).
\]

We have therefore, since \( \psi \) is zero for \( x_1 x_2 \geq 2\delta(\varepsilon) \),
\[
|v_i \psi_2|_{L^2(\Omega)} \leq c \left\{ \frac{1}{x_1} \left| \frac{v_i}{x_1} \right|_{L^2(\Omega)} + \left( \int_{x_1 x_2 \leq 2\delta(\varepsilon)} |D_1 \zeta|^2 dx \right)^{1/2} \right\}. \] (108)

Let us take care of the first term. If \( v_i \in H^1_0(\Omega) \) then \( v_i(\cdot, x_2) \in H^1_0(\mathbb{R}) \) for almost all \( x_2 \) (define \( v_i \) to be zero for \( (x_1, x_2) \) outside \( \Omega \)). Therefore
\[
\left| \frac{v_i}{x_1} \right|_{L^2(\Omega)} = \int \left( \frac{v_i(x_1, x_2)}{x_1} \right)^2 dx_1 dx_2 \\
\quad = \int dx_2 \int_0^{\infty} \left( \frac{v_i(x_1, x_2)}{x_1} \right)^2 dx_1 \leq \text{(by (107))} \\
\quad \leq 2 \int dx_2 \left| D_1 v_i(x_1, x_2) \right|^2 dx_1 \\
\quad = \text{(Fubini again)} \\
\quad = 2 \int \left| D_1 v_i(x_1, x_2) \right|^2 dx_1 dx_2 \\
\quad \leq 2 \|v_i\|_{H^1_0(\Omega)}.
\]

The computation for \( |v_i/x_2|_{L^2(\Omega)} \) is similar and we have
\[
\left| \frac{v_i}{x_j} \right|_{L^2(\Omega)} \leq c\|v\|. \] (109)

For the second term, using Hölder’s inequality and denoting by \( C \) a constant that possibly change in each line, we see that
\[
\left( \int_{x_1 x_2 \leq 2\delta(\varepsilon)} v_i^2 |D_j \zeta|^2 dx \right)^{1/2} \leq d(\varepsilon) \|v\|_{L^4} \leq (\text{thanks to (18)}) \leq C d(\varepsilon) \|v\| \] (110)

where
\[
d(\varepsilon) = \left( \int_{x_1 x_2 \leq 2\delta(\varepsilon)} |D_j \zeta|^4 dx \right)^{1/4}.
\]
By the Lebesgue Dominated Convergence Theorem, \( d(\varepsilon) \) goes to zero when \( \varepsilon \) goes to zero. Hence, since the computation for \( \psi_1 \) is similar, using (108), (109) and (110) we have
\[
|v_t \psi_j|_{L^2(\Omega)} \leq C(\varepsilon + d(\varepsilon))\|v\| \leq C(\varepsilon + \delta(\varepsilon)^{1/4})\|v\|.
\]
Therefore
\[
\|b(v, \psi, v)\| = \|b(v, v, \psi)\| \leq \|v\| \left(\sum_{i,j=1}^{2} |v_t \psi_j|_{L^2(\Omega)}\right) \leq C(\varepsilon + \delta(\varepsilon)^{1/4})\|v\|^2.
\]

Thus, for \( \varepsilon \) sufficiently small, condition (31) is satisfied. Moreover, since \( \psi = u_0 \) belongs to \( \mathcal{C}^\infty(\Omega) \) with divergence zero, \( \nabla u_0 \in L^2(\Omega) \) and \( u_5 \in L^\infty(\Omega) \). Hence, theorems 2 and 3 hold in this case.

\[Q.E.D.\]

### 5.4 Infinite Channel with an Irregular Obstacle

Let us now consider an infinite channel with an irregular obstacle like it is shown in Figure 2. We assume that \( u = U \bar{c}_1 \) and \( T = T_0 \), constants, on both sides of the channel. We could assume that they are different in a bounded part of the channel, but for simplicity we assume not.

\[
\begin{array}{c}
T=T_0 \\
u=U \bar{c}_1
\end{array}
\]

\[
\begin{array}{c}
T=T_1 \\
u=0
\end{array}
\]

Figure 2: Infinite-Channel Flow

We assume that the irregular obstacle has a positive distance from the boundary of the channel, a no-slip boundary condition for the velocity and that the temperature \( T = T_1 \), a constant, on the obstacle.

Since the obstacle has a positive distance from the boundary, we can obtain \( \Omega_1 \) and \( \Omega_2 \), open, smooth, bounded and connected, \( \bar{\Omega}_1 \subset \bar{\Omega}_2 \subset \Omega \), such that the obstacle is contained in \( \Omega_1 \). Let \( (\phi_u, \phi_T) = (0, T_1) \) on \( \partial \Omega_1 \) and \( (\phi_u, \phi_T) = (U \bar{c}_1, T_0) \) on \( \partial \Omega_2 \). Using Section 5.1, we can construct \( z_6 \) on \( \Omega_2 \setminus \Omega_1 \). Extend it to \( \Omega \) by setting \( z_6 = (U \bar{c}_1, T_0) \) on \( \Omega \setminus \Omega_2 \) and \( z_6 = (0, T_1) \) on \( \Omega \cap \Omega_1 \). Note that since \( u_5 \cdot n \) is not zero on \( \partial \Omega \), we cannot apply Section 5.2 to construct \( z_6 \).

It is easy to check that \( z_6 \) will satisfy the boundary conditions, (23)–(29), (31) and (32). Therefore we can apply Theorems 2 and 3 here.

### 5.5 Multi-Channel Flow

Finally we consider a multi-channel flow like is shown in Figure 3. We assume that \( u_p \) is the Poiseuille flow on the input of the main channel and on each output. We assume that the temperature is constant on the boundary.

We can construct \( z_6 \) satisfying the boundary conditions above applying Section 5.1 for an appropriately chosen subdomain in a similar way that was done for the flow in a channel with irregularities (Section 5.3).

**Acknowledgment.** This work was partially supported by the National Science Foundation under the Grant NSF-DMS0074334 and by the Research Fund of Indiana University. The first author was also supported by a scholarship from CNPq-Brasília, Brazil. The second author was partially supported by CNPq-Brasília, Brazil.
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