Regularity issues related to the asymptotic behavior of solutions of the Navier-Stokes equations

Ricardo M. S. Rosa
Instituto de Matemática
Universidade Federal do Rio de Janeiro
(IM-UFRJ)

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3D Navier-Stokes (for homogeneous incompressible fluids)

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \nu \Delta \mathbf{u} + \mathbf{f}, \\
\nabla \cdot \mathbf{u} &= 0.
\end{align*}
\]

\( \mathbf{u} = (u_1, u_2, u_3) = \) velocity field,  
\( \mathbf{x} = (x_1, x_2, x_3) = \) space variables,  
\( t = \) time variable,  
\( p = \) kinematic pressure,  
\( \mathbf{f} = (f_1, f_2, f_3) = \) density of volume forces,  
\( \nu = \) kinematic viscosity.
Known results

- Global (in time) existence of weak solutions not necessarily smooth or unique
- Local (in time) existence of smooth unique solutions not necessarily global
- A bit of regularity \((L^s(0,T;L^r(\Omega)^3), r > 3, \frac{2}{s} + \frac{3}{r} \leq 1)\) implies smooth unique solutions
- “Fractal” estimates for the singularity set (”\(\nabla \otimes u = \infty\)” in time, \(d_H(S_t) \leq 1/2\) (Hausdorff dimension), and in space-time, \(P_1(S_{e:t}) = 0\) (Hausdorff parabolic measure – ”time counts twice”)
- Global well-posedness is one of the US$1,000,000-prizes of the Clay Foundation.
Weak solutions

- Lack of uniqueness $\Rightarrow$ cannot define semigroup in classical sense.

- Yet several dynamical system concepts may be adapted: attractors, bifurcations, invariant measures, etc.
New asymptotic regularity results

- If the weak-limit-set of a weak solution is made of continuous functions with values in $H$ then the weak-limit-set is strongly attracting.

- Time-average stationary statistical solutions of the 3D NSE are partly carried by sets of smooth solutions.
Regular weak limit sets are strongly attracting

- Let $u \in C([0, \infty), H_w)$ be a global weak solution, where $H_w = \text{the space } H \text{ endowed with the weak topology},$ and $H = \text{the space of } L^2 \text{ divergence-free vector fields with appropriate boundary conditions}.$

- The weak-limit-set $\omega_w(u) = \{v_0; u(t_j) \rightharpoonup v_0, t_j \to \infty\}$ is nonempty, weakly compact, somehow invariant, and weakly attracts $u.$

- **Main result:** If $u(t_j + \cdot)$ converges weakly to $v(\cdot)$ and $v$ is continuous from the left at $t = 0$ in $H,$ then the convergence is strong.

- Implies regular weak-limit-sets (fixed points, limit cycles, weak attractors, etc.) are strongly attracting.
Idea of the proof

- We know $u(t_j) \rightharpoonup v_0$ by the definition of $\omega_w(u)$.
- Thus, $|v_0| \leq \liminf_j |u(t_j)|$.
- Use the energy inequality to show $\limsup_j |u_j|^2 \leq |v_0|^2$.

\[
\frac{1}{2} |v_0|^2 = \lim_{\tau_k \to 0^-} \left\{ \frac{1}{2} |v(\tau_k)|^2 - \int_0^{\tau_k} (f, v(s)) \, ds \right\} \\
= \lim_{\tau_k \to 0^-} \lim_{j \to \infty} \left\{ \frac{1}{2} |u(t_j + \tau_k)|^2 - \int_0^{\tau_k} (f, u(t_j + s)) \, ds \right\} \\
\geq \limsup_{j \to \infty} \frac{1}{2} |u(t_j)|^2.
\]

- Then, $|u(t_j)| \to |v_0|$ and $u(t_j) \rightharpoonup v_0$, which implies the strong convergence $u(t_j) \to v_0$. 

Asymptotic partial regularity for time averages

- Turbulent flows have well-defined statistical properties.
- Individual solutions are unpredictable but averages are well-behaved.
- Averages are associated with probability measures.
- Stationary turbulence is associated with probability measures called stationary statistical solutions, a generalization of the notion of invariant measure.
- Some stationary statistical solutions can be obtained as the (generalized) limit of time averages of individual solutions (akin to ergodic theory).
- Are these time-average stationary statistical solutions regular in the sense of being carried, or even supported, in sets more regular than $H$?
**Turbulent flows:** several length scales active, unpredictable, but well-behaved in a statistical sense.

*Figure 1.3  Instantaneous and time averaged views of a jet in cross flow. The jet exits from the wall at left into a stream flowing from bottom to top (Su & Mungal, 1999).*
Types of averages:

Time average: \( \bar{U}(x) \approx \frac{1}{T} \int_0^T u(t, x) \, dt \)

Ensemble average: \( \bar{U}(x) \approx \frac{1}{N} \sum_{n=1}^N u^{(n)}(t, x) \)

Space average: \( \bar{U}(x) \approx \frac{1}{N} \sum_{n=1}^N u(t, x + \ell^{(n)}) \)

**Ergodic hypothesis made in the conventional theory of turbulence:** The mean values are independent of the type of average considered.
Measures related to ensemble averages

- At each time $t$, there is a probability measure $\mu_t$ for the distribution of the velocity field of the flow.

- The statistical information is contained in $\mu_t$. The generalized moments are

$$\langle \varphi(u) \rangle = \int_H \varphi(v) \, d\mu_t(v)$$

which contain, in particular, the classical moments

$$\varphi(u) = (u - \langle u \rangle)^k.$$ 

- Which are the relevant measures for the flow?

- Is there an equation for $\mu_t$?
Evolution of the generalized moments

- For $N$ flows

$$\frac{d}{dt} \langle \varphi(u(t)) \rangle = \frac{d}{dt} \frac{1}{N} \sum_{n=1}^{N} \varphi(u^{(n)}(t)) = \frac{1}{N} \sum_{n=1}^{N} \frac{d}{dt} \varphi(u^{(n)}(t))$$

$$= \frac{1}{N} \sum_{n=1}^{N} \varphi'(u^{(n)}(t)) \circ \frac{d}{dt} u^{(n)}(t)$$

$$= \frac{1}{N} \sum_{n=1}^{N} \varphi'(u^{(n)}(t))) \circ F(u^{(n)}(t))$$

$$= \frac{1}{N} \sum_{n=1}^{N} \langle F(u^{(n)}(t)), \varphi'(u^{(n)}(t))) \rangle_{V',V}.$$

- More generally, for a measure $\mu_t$ in $H$,

$$\frac{d}{dt} \int_{H} \varphi(v) \, d\mu_t(v) = \int_{H} (F(v), \varphi'(v)) \, d\mu_t(v)$$
The formulation

$$\frac{d}{dt} \int_{H} \varphi(v) \, d\mu_t(v) = \int_{H} (F(v), \varphi'(v)) \, d\mu_t(v)$$

avoids the explicit dependence on the individual solutions of the NSE, introducing the dummy variable $v$, with unknown $\mu_t$.

This is a Liouville-type equation, called the statistical Navier-Stokes equations.

The term $F(u) = f - \nu Au - B(u, u)$ “lives” in the dual space $V'$, so only moments with $\varphi'(v)$ in $V$ a.e. can be considered.
Turbulence in statistical equilibrium in time

- A flow is in *statistical equilibrium* in time when the generalized moments are independent of $t$:

$$\int_{H} \varphi(u) \, d\mu_t(v) = \text{independent of } t.$$  

- Ensemble averages of flows in statistical equilibrium in time lead to the notion of *stationary statistical solution* $\mu_t \equiv \mu$.  

Stationary statistical solutions

Borel probability measure $\mu$ in $H$ satisfying

- Finite mean kinetic energy: $\int_H |\mathbf{v}|^2 \, d\mu(\mathbf{v}) < \infty$;
- Finite mean enstrophy: $\int_H |\nabla \otimes \mathbf{v}|^2 \, d\mu(\mathbf{v}) < \infty$;
- Energy inequality

$$\int_{\{e_1 \leq \frac{1}{2} |\mathbf{v}|^2 < e_2\}} \{\nu \|\mathbf{v}\|^2 - (\mathbf{f}, \mathbf{v})\} \, d\mu(\mathbf{v}) \leq 0,$$

for all energy levels $0 \leq e_1 \leq e_2 \leq \infty$;

- Stationary statistical Navier-Stokes equations

$$\int_H (\mathbf{F}(\mathbf{v}), \Phi'(\mathbf{v})) \, d\mu(\mathbf{v}) = 0,$$

for all suitable test functions.
Stationary statistical solutions and time averages

- Let \( u = u(t), t \geq 0 \), be a weak solution and let \( \varphi \in C(H_w) \).

- Then \( \varphi(u(t)) \) is bounded in \( t \geq 0 \), just like
  \[
  (0, \infty) \ni T \mapsto \frac{1}{T} \int_0^T \varphi(u(t)) \, dt.
  \]

- A generalized limit defines a positive linear funcional on \( C(H_w) \), with \( H_w \) locally compact:
  \[
  \varphi \mapsto \lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(u(t)) \, dt.
  \]

- Kakutani-Riesz Representation Theorem: there exists a Borel measure \( \mu = \mu_u \) on \( H_w \) such that
  \[
  \lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(u(t)) \, dt = \int_H \varphi(v) \, d\mu_u(v).
  \]
Regularity of time-average statistical solutions

- Let \( u = u(t) \) be a weak solution and let \( \mu_u \) be the associated SSS:

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(u(t)) \, dt = \int_H \varphi(v) \, d\mu(v).
\]

- Let \( \omega_w(u) \) be the weak-limit-set of \( u \) in \( H \).
- Then \( \text{supp} \ (\mu_u) \subset \omega_w(u) \).
- \( \mu_u(V) = 1, \mu_u(D(A)) = 1 \).
- Recurrence: Sets \( E \subset H \) with \( \mu_u(E) > 0 \) are “recurrent”.
- Asymptotic regularity question: Is \( \mu_u \) supported or carried on a more regular set? On a set where global regularity holds?
Asymptotic regularity

\[ A_w = \left\{ \mathbf{u}_0 \in H; \exists t_0 \in \mathbb{R} \text{ and a bounded global solution } \mathbf{u} \text{ with } \mathbf{u}(t_0) = \mathbf{u}_0 \right\} \]

\[ A_{\text{reg}}' = \left\{ \mathbf{u}_0 \in H; \exists \delta > 0, t_0 \in \mathbb{R} \text{ and a bounded global solution } \mathbf{u} \text{ with } \mathbf{u}(t_0) = \mathbf{u}_0 \text{ and } \mathbf{u} \text{ regular on } (t_0 - \delta, t_0 + \delta) \right\} \]

\[ A_{\text{reg}}^\infty = \left\{ \mathbf{u}_0 \in H; \exists t_0 \in \mathbb{R} \text{ and a bounded global regular solution } \mathbf{u} \text{ with } \mathbf{u}(t_0) = \mathbf{u}_0 \text{ and } \mathbf{u} \text{ unique among bounded global solutions} \right\} \]

- We have \( \mu_{\mathbf{u}}(A_w) = 1 \), where \( A_w = \) weak global attractor

- We have \( \mu_{\mathbf{u}}(A'_{\text{reg}}) > 1/(1 + n) \), for \( n \) large enough (related to the fractal estimate of the set of singularities in time), with \( A'_{\text{reg}} \) open and dense in \( A_w \)

- We look for \( \mu_{\mathbf{u}}(A_{\text{reg}}^\infty) = 1 \)