ON THE LANDAU-LIFSCHITZ DEGREES
OF FREEDOM IN 2-D TURBULENCE

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Abstract. We show that if the Kraichnan theory of fully developed turbulence holds, then
the Landau-Lifschitz degrees of freedom is bounded (up to a logarithmic term) by $G^{1/2}$, where
$G$ is the Grashof number.

The incompressible Navier-Stokes equations (NSE) with periodic boundary conditions
on $[0, L]^2$ can be written as

\begin{equation}
\frac{du}{dt} + \nu Au + B(u, u) = f
\end{equation}

where $A = -\Delta$, $B(u, v) = \mathcal{P}((u \cdot \nabla)v)$ with $\mathcal{P}$ the Helmholtz-Leray projection onto diver-
gence free functions, and $f$ is a body force. We assume that $f = P_{\kappa_0}f$, where

$$P_{\kappa}u = \sum_{\kappa_0 |k| \leq \kappa} \hat{u}_k e^{i\kappa_0 k \cdot x}, \quad \text{for } u(x) = \sum_{k \in \mathbb{Z}^2} \hat{u}_k e^{i\kappa_0 k \cdot x}, \quad \text{with } \kappa_0 = 2\pi/L,$$

and that $\kappa_0 / \kappa_0 \leq C_0$. Critical wave numbers $\kappa_\eta, \kappa_\sigma$, are defined through the generalized
time averages (see [FJMR])

\begin{equation}
\eta = \frac{\nu}{L^2} \langle |A|^2 \rangle, \quad \epsilon = \frac{\nu}{L^2} \langle |A^{1/2}u|^2 \rangle \quad \text{as } \kappa_\eta = \left( \frac{\eta}{L^3} \right)^{1/6} \quad \text{and } \kappa_\sigma = \left( \frac{\eta}{\epsilon} \right)^{1/2},
\end{equation}

where $| \cdot |$ is the $L^2$-norm.

It is shown in [FJMR] that if the Kraichnan theory of fully developed turbulence [K67]
holds for the NSE, then

\begin{equation}
\left( \frac{\kappa_\eta}{\kappa_0} \right)^2 \leq \left( \frac{1}{2\pi} \right)^{2/3} \left[ \left( \frac{\kappa_\sigma}{\kappa_0} \right)^2 - 1 \right]^{-1/3} G^{2/3},
\end{equation}

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where $G = |f|/(\nu \kappa_0)^2$ is the Grashof number. The ratio $(\kappa_\eta/\kappa_0)^2$ is the Landau-Lifschitz asymptotic degrees of freedom, which is shown in [CFM94] to be an upper bound on $\dim F(A)$, the fractal dimension of the global attractor [T] (up to a logarithmic term in $(\kappa_\eta/\kappa_0)$). We also show in [FJMR] that if the Kraichnan theory holds, then

$$
(1.4) \quad \kappa_\sigma \sim \kappa_\eta \left( \ln \frac{\kappa_\eta}{\kappa_i} \right)^{-1/2},
$$

where $\kappa_i$ is the lower endpoint of the inertial range. Using (1.4) in (1.3) leads in [FJMR] to the somewhat surprising estimate $(\kappa_\eta/\kappa_0)^2 \lesssim G^{4/7}$ (up to a logarithmic term). This undercuts the previous best estimate $(\kappa_\eta/\kappa_0)^2 \lesssim G^{2/3}$ (up to a logarithmic term), made in [CFM94] without assuming turbulence.

The power $4/7$ does not, however, fully exploit the relations (1.4) and (1.3). In fact, we show in the next few lines that $(\kappa_\eta/\kappa_0)^2 \lesssim G^{1/2}$ (up to a logarithmic term).

Use (1.4) in (1.3) to obtain

$$
\left( \frac{\kappa_\eta}{\kappa_0} \right)^6 \left[ \left( \frac{\kappa_\eta}{\kappa_0} \right)^2 \left( \frac{\kappa_0}{\kappa} \right)^2 \left( \ln \frac{\kappa_\eta}{\kappa_i} \right)^{-1} - 1 \right] \lesssim G^2.
$$

Apply the estimate $\kappa_\eta/\kappa_0 \leq G^{1/3}$ from [FMT93] to reach

$$
\left( \frac{\kappa_\eta}{\kappa_0} \right)^8 \left( \frac{\kappa_0}{\kappa} \right)^2 \left( \ln \frac{\kappa_\eta}{\kappa_i} \right)^{-1} \lesssim G^2 + \left( \frac{\kappa_\eta}{\kappa_0} \right)^6 \leq 2G^2,
$$

from which immediately follows

$$
(1.5) \quad \left( \frac{\kappa_\eta}{\kappa_0} \right)^2 \left( \ln \frac{\kappa_\eta}{\kappa_i} \right)^{-1/4} \lesssim \left( \frac{\kappa}{\kappa_0} \right)^{1/2} G^{1/2}.
$$

REFERENCES


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