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Introduction to Hamiltonian Perturbation Theory

and Nekhoroshev Theory
Step by step

- Neatly integrable Hamiltonian systems
- Integrable Hamiltonian systems with 2 or more DOF
- A starting point: one degree of freedom systems
- Integrable and quasi-integrable Hamiltonian systems
- Preliminary theory
- Perturbation construction and technical tools
- Nekhoroshev theory
- One perturbation step
- The averaging principle
- Hamiltonian perturbation theory
- Stability results
- Perturbation construction
- Preliminary theory
Integrable and quasi-integrable Hamiltonian systems

A starting point: one degree of freedom systems

Hamiltontian perturbation theory

Near integrable Hamiltonian systems with 2 or more DOF

Nearly integrable Hamiltonian systems with finitely many Fourier components

Perturbation construction

Stability results

Nekhoroshev theory

The averaging principle

Perturbation with finitely many Fourier components

Preliminaries and technical tools

Hamiltonian perturbation theory

Step by step
Near-integrable and quasianitegrable Hamiltonian systems
Hamiltonian perturbation theory
Nekhoroshev theory

Action-angle variables for the pendulum

- \( H(p,q) = \frac{p^2}{2} - \omega \cos q \)
- \( \phi = \partial S / \partial I = \partial S / \partial h \)
- \( I(t) = I_0 \), \( \phi(t) = \phi_0 + \omega t \)

\( \oint Mh \, dp \, dq = \pi \int q + (h) \sqrt{f(h + \omega \cos x)} \, dx \)

The Hamiltonian in the new variables \((I, \phi)\) is

\( H(I, \phi) = H(I) \)

Motions are simply

\( I(t) = I_0 \), \( \phi(t) = \phi_0 + \omega t \)
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fiamiltonian perturbation theory
Nekhoroshev theory
u starting point
n one degree of freedom
sy-stems

flntegrable and quasiaintegrable fiamiltonian systems

fiamiltonian perturbation theory

Nekhoroshev theory

u starting point

n one degree of freedom

sy-stems

\[ H(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T \mathbf{f} - \mathbf{f}^T \mathbf{w} \]

\[ \mathbf{H}(\mathbf{I}, \mathbf{\theta}) = \mathbf{I} \]

\[ S(\mathbf{I}, \mathbf{\theta}) = \int \mathbf{\theta} d\mathbf{\phi} \]

\[ \phi = \frac{\partial S}{\partial \mathbf{I}} = \frac{\partial S}{\partial \mathbf{\theta}} \]

\[ \mathbf{H}^\prime(\mathbf{I}, \mathbf{\theta}) = \mathbf{H}(\mathbf{I}, \mathbf{\theta}) \]

\[ \mathbf{I}(\mathbf{t}) = \mathbf{I}_0, \quad \mathbf{\theta}(\mathbf{t}) = \mathbf{\theta}_0 + \mathbf{\omega} \mathbf{t} \]

Introduction to Hamiltonian Perturbation Theory

Action-angle variables for the pendulum
The Hamiltonian in the new variables is
\[ H'(I, \phi) = H(I) \]
Motion are simply
\[ (I)\dot{y} = (\phi \cdot I) \cdot H \]
\[ \dot{y} = (b \cdot d) H \]
\[ b \cos x - \frac{x}{2} = (b \cdot d) H \]

Action-angle variables for the pendulum
The Hamiltonian in the new variables is
\[ H(p, q) = \frac{p^2}{2m} + V(q) \]
where
\[ V(q) = \frac{1}{2}m \omega^2 q^2 \]
and
\[ \mathbf{p} = (p_x, p_y), \quad \mathbf{q} = (q_x, q_y) \]
where
\[ (p_x, p_y) \]
are the canonical momenta and
\[ (q_x, q_y) \]
are the coordinates.

The Hamiltonian for the pendulum is
\[ H(p, q) = \frac{p^2}{2m} + mg \sin q \]
where
\[ (p, q) \]
are the canonical variables and
\[ (p, q) \]
are the coordinates.

The Hamilton-Jacobi equation is
\[ H(\mathbf{P}, \mathbf{Q}) = \int S(\mathbf{P}, \mathbf{Q}) \sqrt{g(\mathbf{P}, \mathbf{Q})} \, d\mathbf{P} \]
where
\[ g(\mathbf{P}, \mathbf{Q}) = \sum_{i,j} g_{ij} \partial_i S \partial_j S \]
are the metric coefficients.

The action-angle variables are
\[ (I, \phi) \]
where
\[ I = \int p \, dq \]
and
\[ \phi = \int \sqrt{g} \, dq \]
are the action and angular momentum, respectively.

The equations of motion are
\[ \dot{I} = \frac{\partial H}{\partial \phi} = 0 \]
and
\[ \dot{\phi} = \frac{\partial H}{\partial I} = 0 \]
which are satisfied by
\[ I = I_0 \]
and
\[ \phi = \phi_0 \]
for some constants
\[ I_0 \]
and
\[ \phi_0 \].


\[ H(p, q) = p f - \omega f \cos q \]

We fix \( H(p, q) = \hbar I(h) = e^{\pi \oint M h p \, dq} = e^{\pi \int q + (\hbar) q - (\hbar) \sqrt{f(h + \omega f \cos x)}} \, dx \)

\[ S(h, q) = \int q \, d\sqrt{f(h + \omega f \cos x)} \, dx, \]

\[ \phi = \partial S / \partial I \]

\[ \phi = \partial S / \partial h \cdot dh \cdot dI \]

The \( \hbar \) (new variables is)

The Hamiltonian in the new variables

\[ \frac{\partial H}{\partial \phi} = \frac{\partial H}{\partial \hbar} = \phi \]

\[ x p(x \cos \varphi + \Psi) \sqrt{\frac{(u) - b f - \frac{z}{2}}{1}} = (b' \Psi) \]

\[ x^2 \cos \varphi - \frac{z^2}{2} = (b' d) H \]
\[ H(p, q) = \frac{1}{2} p^2 - \omega_p \cos q \]

We fix
\[ H(p, q) = h(I) = e^{\pi \oint M_i p \, dq} = e^{\pi \int q + (h) q - (h) \sqrt{f(h + \omega f \cos x)} \, dx} \]

\[ S(h, q) = \int q d\sqrt{f(h + \omega f \cos x)} \]

\[ \phi = \partial S / \partial I = \partial S / \partial h \, dh \]

The Hamiltonian in the new variables is
\[ (\phi, I) \]

Motions are simply
\[ \dot{I} = (\phi, I) H \]

\[ \dot{\phi} = (\phi, I) \frac{\partial H}{\partial I} \]

\[ f(x \cos z + y) \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} \]

\[ = b \phi d \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} = (y) \]

\[ h = (b \cdot d) H \]

\[ b \cos z = (b \cdot d) H \]

Action-angle variables for the pendulum
They are always integrable (i.e., one can construct action-angle variables as for the pendulum) systems and not because they make equations of motion simpler. Action-angle variables are useful to deal with perturbed systems and not because they make equations of motion simpler; one can study it by means of perturbation theory.

Example:

\[ H(p,q,J,\psi) = \frac{p^2}{2} - \omega q \cos q + \frac{\epsilon}{2} + \epsilon V(q,\psi) \]

\[ K(I,J,\phi,\psi) = H(I) + \frac{\epsilon}{2} + \epsilon f(I,\phi,\psi) \]

Near 

One degree of freedom Hamiltonian systems

Integrable and Quasiintegrable Hamiltonian Systems

Nekhoroshev's theory

Starting point: one degree of freedom (one can study it by means of perturbation theory)

Example:

\[ H(p,q,J,\psi) = \frac{p^2}{2} - \omega q \cos q + \frac{\epsilon}{2} + \epsilon V(q,\psi) \]

\[ K(I,J,\phi,\psi) = H(I) + \frac{\epsilon}{2} + \epsilon f(I,\phi,\psi) \]
One degree of freedom Hamiltonian systems

Example:

Action-angle variables are useful to deal with perturbed systems and not because they make equations of motion simpler, but because they make equations of motion integrable (i.e., one can construct

\[ H(q, p, \lambda, \mu) = pQ + \mu \]
One degree of freedom Hamiltonian systems

is difficult to deal with. (§)

\[(\dot{\phi} \cdot \dot{j} \cdot j \cdot j + \frac{\partial}{\partial j}) \cdot \phi = (\dot{\phi} \cdot \dot{j} \cdot j \cdot j \cdot j) \cdot \phi \]

\[ (\dot{\phi} \cdot \dot{j} \cdot \phi \cdot \dot{j} \cdot j + \frac{\partial}{\partial j}) \cdot \phi = (\dot{\phi} \cdot \dot{j} \cdot \phi \cdot \dot{j} \cdot j \cdot j) \cdot \phi \]

Example: Ø

simpler.

Action-angle variables are useful to deal with perturbed action-angle variables as for the pendulum.

They are always integrable (i.e. one can construct

Near starting point in one degree of freedom Hamiltonian systems

action-angle variables as for the pendulum.)

Near starting point in one degree of freedom Hamiltonian systems

Near starting point in one degree of freedom Hamiltonian systems
Neatly integrable Hamiltonian systems

Nearly integrable Hamiltonian systems with 2 or more DOF

A starting point: one degree of freedom systems

Integrable and quas-integrable Hamiltonian systems

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Stability results
and we restrict our attention to it. We assume compact (or containing a compact component)
which is therefore an n-dimensional submanifold of $D$, which
$$u = \begin{pmatrix} F_1(p_1, q_1) \\ \vdots \\ F_n(p_n, q_n) \end{pmatrix}$$
rank $F_1, \ldots, F_n$ are linearly independent on $D$, i.e.
$$\mathcal{F} \ni (c_1, \ldots, c_n) \mapsto \{u_1, \ldots, u_n \in \mathbb{R}^n : F_i(u) = c_i, i = 1, \ldots, n \} = \mathcal{C}$$
with canonical coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n) = (b, d)$.
We consider a chart of the phase space $D$ open.
$$f = \{f_1, \ldots, f_n \in \mathbb{R} : f_i \neq 0 \} \text{ with } \mathcal{C} \cap f = \emptyset.$$
and we restrict our attention to it. we assume compact (or containing a compact component)
is therefore an n-dimensional submanifold of D, which

\[ u = \left[ \begin{array}{c} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{array} \right] \]

rank

\[ \psi_i \text{ are linearly independent on } \mathbb{R}^n \text{.} \]

\[ \{ \psi_i \} \subseteq \{ \phi_i \} \text{ } \psi_i \in \mathbb{C} \text{ and } \psi_i \text{ are linearly independent on } \mathbb{R}^n \text{.} \]

\[ \{ \psi_i \} = \{ \phi_i \} \]

\[ \psi_i \in \mathbb{C} \text{ and } \psi_i \text{ are linearly independent on } \mathbb{R}^n \text{.} \]

We consider a chart of the phase space \( \mathbb{R}^n \).

\[ \Psi \in \mathbb{R}^n \]

The Hamiltonian flows pairwise commute:

\[ \{ F_i , F_j \} = 0 \]

If it has no first integrals, then we assume that

freedom and we assume that

We consider an Hamiltonian system with n-degrees of

Generalization of the previous case
and we restrict our attention to it. We assume compact (or contractible) a compact component.

$u = \left( \frac{\partial F_i}{\partial p}, \ldots, \frac{\partial F_i}{\partial p_n}, \frac{\partial F_i}{\partial q}, \ldots, \frac{\partial F_i}{\partial q_n} \right)$

Thus is an $n$-dimensional submanifold of $D$, which is linearly independent on $D$. Let $u \in \mathbb{R}^n$ be an $n$-tuple.

$\{u \in \mathbb{R}^n : F_i(u) = c, i = 1, \ldots, n \}$

is a submanifold of $D$ with compact codimension $n$. We consider a chart of the phase space $D$ open.

$F_i(u) = 0$ for $i = 1, \ldots, n$

They Hamiltonian flows pairwise commute.

If has $n$ first integrals, they commute.

We consider an Hamiltonian system with $n$-degrees of freedom and we assume that $n$-independent first integrals.

**Generalization of the previous case**
We consider an Hamiltonian system with $n$-degrees of freedom and we assume that $q$ is time dependent Hamiltonian systems.

We assume compact (or contains a compact component) $q$ is then is an $n$-dimensional submanifold of $G$, which

\[ u = \left( \begin{array}{c} u_1 \\ \vdots \\ u_n \end{array} \right) \quad \text{rank} \quad \left( \begin{array}{cccc} F_{e_1} & \cdots & F_{e_n} \\ \vdots & & \vdots \\ F_{n_e_1} & \cdots & F_{n_e_n} \end{array} \right) = n \]

\[ \wedge \in \{ (\omega_1, \ldots, \omega_n) \in (b, d) \mid \omega_i \in (b, d) \} \]

{
- We consider a chart of the phase space: $\mathbb{R}^{2n}$ open
- Their Hamiltonian flows pairwise commute.
- It has $n$ first integrals $F_1, \ldots, F_n$.
- We consider an Hamiltonian system with $n$-degrees of freedom and we assume that $q$ is time dependent Hamiltonian systems.

Generalization of the previous case

We consider compact (or contains a compact component) $q$ is then is an $n$-dimensional submanifold of $G$, which

\[ u = \left( \begin{array}{c} u_1 \\ \vdots \\ u_n \end{array} \right) \quad \text{rank} \quad \left( \begin{array}{cccc} F_{e_1} & \cdots & F_{e_n} \\ \vdots & & \vdots \\ F_{n_e_1} & \cdots & F_{n_e_n} \end{array} \right) = n \]

\[ \wedge \in \{ (\omega_1, \ldots, \omega_n) \in (b, d) \mid \omega_i \in (b, d) \} \]
We consider an \( n \)-dimensional phase space with \( n \)-degrees of freedom and we assume that the Hamiltonian flows pairwise commute. The Hamiltonian flows pairwise commute.

If there is at least one integral, \( H \in \mathbb{R}^n \), \( \mathbb{R}^n \) is integrable, and we assume that

\[
\{H, \{b_1, \ldots, b_n, d_1, \ldots, d_n\}\} = 0.
\]

We consider a chart of the phase space: \( C \subset \mathbb{R}^n \) open.

The Hamiltonian flows pairwise commute.

We consider an integrable Hamiltonian system with \( n \)-degrees of freedom.
We consider an integrable Hamiltonian system with $n$-degrees of freedom and we assume that the system is linearly independent on a compact component of $\mathbb{R}^n$. Let $u = \left( (b_1, \ldots, b_n) \right) \in \mathbb{R}^n$ with canonical coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbb{R}^{2n}$.

Their Hamiltonian flows pairwise commute:

$$\{u, \ldots, v\} = \{ u, v \} = 0, \quad \forall u, v \in \mathbb{R}^n.$$ 

We consider an integrable Hamiltonian system with $n$-degrees of freedom and we assume that the system is linearly independent on a compact component of $\mathbb{R}^n$. Let $u = \left( (b_1, \ldots, b_n) \right) \in \mathbb{R}^n$ with canonical coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbb{R}^{2n}$.

The Hamiltonian flows pairwise commute:

$$\{u, \ldots, v\} = \{ u, v \} = 0, \quad \forall u, v \in \mathbb{R}^n.$$
and we restrict our attention to it. We assume compact (or contains a compact component) is therefore an n-dimensional submanifold of \( D \), which

\[
u = \begin{pmatrix}
\nu_1 \\
\nu_2 \\
\vdots \\
\nu_n
\end{pmatrix}
\]

\( \text{Rank} \) of \( \nu \) are linearly independent on \( \mathbb{R}^n \). Let

\[
\nu \in \begin{pmatrix}
\nu_1 \\
\nu_2 \\
\vdots \\
\nu_n
\end{pmatrix}
\]

\( \{ u, \ldots, \nu_n \} = 1 \Rightarrow (b, d) \in \mathbb{C} \Rightarrow (b, d) \}

with compatible coordinates (b, d) = (b, d) \in \mathbb{C} \Rightarrow (b, d) \}

We consider a chart of the phase space: \( \mathbb{R}^n \) open. \( A \}

\[
\mathbb{A} = \begin{pmatrix}
A_1 \\
A_2 \\
\vdots \\
A_n
\end{pmatrix}
\]

The Hamiltonian flows pairwise commute.

It has n first integrals, \( F_1, \ldots, F_n \)

We consider an Hamiltonian system with n-degrees of freedom and we assume that

\[
F_1, \ldots, F_n
\]
Theorem

If the previous assumptions hold, then:

\[ D_C \neq \emptyset \]

\[ D_C = \bigcup_{c' \in C} \Sigma_{c'} \approx C \times \Sigma_{c'} \]

and in \( D_C \in \mathbb{R}^n \) action-angle variables, \( \omega \in \mathbb{R}^n \) a canonical transformation

\[ (p, q) = w(I, \phi) \]

\[ I \in B \subset \mathbb{R}^n, \quad \phi \in T^n \]

\[ D_C = \bigcup_{c' \in C} \Sigma_{c'} \approx C \times \Sigma_{c'} \]


The Liouville-Arnold theorem
The Liouville-Arnold theorem

If the previous assumptions hold, then:

\[
\Sigma_c \approx T^n \times D^c
\]

\[
D^c = \bigcap_{i \in \mathbb{N}} \mathbb{R}^n
\]

\[
i (I, \phi) = f_i(I), \quad i = e, \ldots, n
\]

where \( \mathbb{R}^n \) is the phase space of the system, and \( I \) are the actions variables. The theorem states that the phase space is foliated by tori, and the action variables are quasi-periodic functions of time.
The Liouville-Arnol'd theorem

If the previous assumptions hold, then:

\[ \mathcal{D} \subset \mathcal{C} \]

and in \( C \in \mathcal{D} \), action-angle variables, i.e., a canonical transformation:

\[ (p, q) \rightarrow (I, \phi) \]

where \( I \in B \subset \mathbb{R}^n \), \( \phi \in T^n \), and

\[ V_{b, \text{bar}}(I) = \mathcal{F}(I) \]

such that: \( I = \text{const} \) on \( T^n \) and conversely:

\[ \mathcal{D} \approx \mathcal{C} \]

and in a neighborhood \( C \) of such that:

\[ \mathcal{D} \subset \mathcal{C} \]

Theorem
Integrable Hamiltonian systems

- The Lagrange top (symmetric rigid body with a fixed point without external torques)
- The Euler rigid body (rigid body with a fixed point without external torques).
- The spherical pendulum.
- A point mass in a central force field in 2 and 3 dimensions.

Example

(continued)

- The rigid body with a fixed point in the gravity.

All Hamiltonian systems to which the Liouville-Arnold theorem applies are called integrable.
In the gravity

- The Lagrange top (symmetric rigid body with a fixed point external torques)
- The Euler rigid body (rigid body with a fixed point without external torques)
- The spherical pendulum
- A point mass in a central force field in 2 and 3 dimensions.

**Example**

All Hamiltonian systems to which the Liouville-Arnold theorem applies are called integrable.
An integrable Hamiltonian system is superintegrable or properly degenerate if \( n^1 < n \).

<table>
<thead>
<tr>
<th>Example</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kepler problem: ( n = 2 ) and ( n^1 = 5 ).</td>
<td></td>
</tr>
<tr>
<td>Euler-Poinsot rigid body: ( n = 3 ) and ( n^1 = 4 ).</td>
<td></td>
</tr>
</tbody>
</table>

A Hamiltonian system is superintegrable or properly degenerate if \( n^1 = \) number of first integrals; \( n = \) degrees of freedom.
Superintegrable Hamiltonian systems

Definition

An integrable system is superintegrable or properly degenerate if

Only if $n > n^1$ actions effectively enter the Hamiltonian.

Motion does not really take place on $\mathbb{R}^n$ but is restricted to

manifolds of dimension $n^1 = 2n - n^0$ (typically they are $\mathbb{S}^n$).

The Kepler problem: $n = 2$ and $n^1 = 5$.

The Euler-Poinsot rigid body: $n = 3$, $n^1 = 4$.

Example

The Kepler problem: $n = 2$ and $n^1 = 5$.

The Euler-Poinsot rigid body: $n = 3$, $n^1 = 4$.

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The Kepler problem: $n = 2$ and $n^1 = 5$.

The Euler-Poinsot rigid body: $n = 3$, $n^1 = 4$.
Only \( \mathfrak{g} > n \) actions effectively enter the Hamiltonian.

By definition of dimension \( n = 2n - n' \) (typically \( n' \approx \mathfrak{g} \)) manifolds of dimension \( n' = 2n - n \) do not really take place on \( \mathbb{W} \) but is restricted to motion does not really take place on \( \mathbb{W} \) but is restricted to \( \mathbb{W} \). An integrable Hamiltonian system is superintegrable or properly degenerate if \( n' < n \).

\[ n' = \text{number of first integrals}; \quad n = \text{degrees of freedom} \]

**Definition of Superintegrable Hamiltonian System**
Example

- The Kepler problem: $n = 3$ and $n! = 5$.
- The Euler-Poinsot rigid body: $n = 3$, $n! = 6$.

Definition

A Hamiltonian system is superintegrable or proper if $n^J < n!$ and $n^J = n$ number of first integrals; $n$ degrees of freedom.

Supersymmetric Hamiltonian systems

- Hamiltonian perturbation theory
- Nekhoroshev theory
- Superintegrable Hamiltonian systems
- Quasiperiodic motion
- The Kepler problem
- The Euler-Poinsot rigid body
- The two-body problem

Nearby and quasi-regular trajectories in $n$ degree of freedom.
What are they?

\[
H = \varepsilon f(I, \varphi) + \mathcal{O}(\varepsilon^0)\]

We know everything about this system

\[
(\phi^* f)(j) \eta = \phi(j) H
\]

One would like to have

\[
(\varepsilon f(I, \varphi) \supset 1 \mathcal{O}(\varepsilon^0) \Leftrightarrow \mathcal{O}(\varepsilon) \supset \mathcal{O}(\varepsilon^0)
\]

Natural estimate (from (1))

Give good upper bounds on the quantity

\[
\lim_{\varepsilon \to 0} (\|I(t) - I_d\|) = d.
\]

Problem

Near integrable Hamiltonian systems if \( \varepsilon > 0 > 1 \): Is behaviour may be very complicated.

Nearly integrable Hamiltonian systems if \( \varepsilon \to 0 \): what are they?
What are they?

Nearly integrable Hamiltonian systems

We know everything about this system

\[ (\phi', \phi)^T + (\phi)H = 0 \]

Give good upper bounds on the quantity

\[ \parallel I(t) - I(0) \parallel \leq C \epsilon t = \Rightarrow \parallel I(t) - I(0) \parallel \leq C' \mathrm{t} \sim o(e/\epsilon) \]

One would like to have

| 2 \parallel I(t) - I(1) \parallel | \sim \parallel I(t) - I(0) \parallel \leq \parallel I(0) - H(1) \parallel |

Integrable estimate (from (1))

- Near integrability: For systems of the form
  - Hamiltonian perturbation theory
  - Nekhoroshev theory
  - Starting point in one degree of freedom systems

\[ V \epsilon \bar{\omega} \text{lin introduction to Hamiltonian perturbation theory} \]
Introduction to Hamiltonian Perturbation Theory

\[ H(\mathbf{I}, \mathbf{\phi}) = h(\mathbf{I}) + \varepsilon f(\mathbf{I}, \mathbf{\phi}) \]

We know everything about this system if \( d < \varepsilon \ll q \) its behaviour may be very complicated.

Near Integrable Hamiltonian Systems

\[ \lim_{t \to 0} \|\mathbf{I}(t) - \mathbf{I}_0\| = 0 \]

One would like to have:

\[ \|\mathbf{I}(t) - \mathbf{I}(0)\| = \varepsilon t \leq \|\mathbf{I}(0) - \mathbf{I}(0)\| \]

Trivial estimate (from (1))

\[ \|\mathbf{I}(t) - \mathbf{I}(0)\| \leq C t = \|\mathbf{I}(0) - \mathbf{I}(0)\| \]

Give good upper bounds on the quantity \( \|\mathbf{I}(t) - \mathbf{I}(0)\| \)

Problem

\[ \lim_{\varepsilon \to 0} \sup_{t \in \mathbb{R}} \|\mathbf{I}(t) - \mathbf{I}_d\| = d. \]

What are they?
Nearly integrable Hamiltonian systems

\[ H(\mathbf{I}, \varphi) = h(\mathbf{I}) + \varepsilon f(\mathbf{I}, \varphi) \]

We know everything about this system.

Problem

Give good upper bounds on the quantity \( \parallel I(t) - I_0 \parallel \)

\[ \parallel I(t) - I_0 \parallel \leq \varepsilon C, \quad t \sim O(e/\varepsilon) \]

One would like to have

\[ \varepsilon^{1/10} \ll \| H(0, \varphi) \| \ll \| H(0, \varphi) \|, \quad \tau \ll \varepsilon \]

\[ \text{Trivial estimate (from (1))} \]

\[ \| H(0, \varphi) \| \ll \varepsilon C \]

What are they?

Nearly integrable Hamiltonian systems

\[ l_{\text{Int}} = \varepsilon^{1/10} \parallel H(0, \varphi) \parallel \]

\[ l_{\text{Ext}} = \varepsilon C \]

Hamiltonian perturbation theory

Nekhoroshev theory

Integrable and quasintegrable Hamiltonian systems
What are they?

One would like to have

\[ (\varepsilon / \hbar) O \sim 1, \quad C \leq \| h - (i) h \| \iff 1 \leq \| h - (i) h \| \]

The initial estimate (from (1))

\[ \| (0) h - (i) h \| \leq \varepsilon C t \]

One would like to have

\[ \lim_{\varepsilon \to 0} \left( \| h - (i) h \| \right) = 0. \]

Letting everything about this system

We know everything about this system

<table>
<thead>
<tr>
<th>Problem</th>
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<tbody>
<tr>
<td>Give good upper bounds on the quantity</td>
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</table>

\[ \sup_{t \in \mathbb{R}} \| I(t) - I_0 \| \leq C', t \sim O(\varepsilon / \varepsilon) \]

\[ H(\phi, \dot{\phi}) = \frac{1}{2} \dot{\phi}^2 + (i) \phi \]

Nearly integrable Hamiltonian systems if

\[ \varepsilon \gg \varepsilon > 0 \]

For the case of nearly integrable systems

\[ \lim_{\varepsilon \to 0} \left( \| I(t) - I_0 \| \right) = d. \]
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fiamiltonian perturbation theory

Nekhoroshev theory

u starting point

one degree of freedom

systems

Sometimes things go wrong...
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Integrable and quasi-integrable Hamiltonian systems
The motion of $I$ on long time scales is reasonable to think that only systematically contribu-
tes to

$$\int_0^\infty \int_0^\infty \frac{\partial}{\partial x} u(x, z) = \langle I \rangle$$

$$(\dot{\phi}, \dot{\psi}) = (\phi, \psi)$$

Slow variables: Fast variables: $H$ is periodic in $\phi$.

$$\frac{\partial \psi}{\partial t} = \frac{\partial \phi}{\partial t} = 0$$

For a nearly integrable Hamiltonian system it will be

$$m \in \mathbb{W} \times \mathbb{W} \ni (\phi, \psi) \quad (\phi, \psi) = \psi$$

$$(\phi, \psi) = \psi$$

Framework

Hamiltonian perturbation theory

The averaging procedure

Nekhoroshev theory

Hamiltonian perturbation theory

Introduction to Hamiltonian perturbation theory

Flntegrable and quasiaflntegrable flntegrable systems

Flntegration

Flntegrable and quasiaflntegrable flntegrable systems

Flntegration

Flntegrable and quasiaflntegrable flntegrable systems

Flntegration
Let: general case
\[ f(0) = (i, r) \]
The averaged system
\[ f(0) = (0, 0) \]
\[ (\forall i > 0) \lim_{\varepsilon \to 0^+} \| I(\varepsilon) r - (i, r) \| = 0 \]

Right: Hamiltonian case
\[ f(0) = (i, r) \]

We want to know if (and for which conditions) the averaged system
\[ f(0) = (0, 0) \]

\[ (\forall i > 0) \lim_{\varepsilon \to 0^+} \| I(\varepsilon) r - (i, r) \| = 0 \]
Let general case

\[ \mathbf{f}(0) = \mathbf{f}(\epsilon) \mathbf{r}(t) \]

Right Hamiltonian case \( \mathbf{S} \]

\[ \mathbf{S} = I(d) \mathbf{T} \]

\[ \mathbf{V} = \begin{pmatrix} \epsilon \\ 0 \end{pmatrix} \]

We want to know if (and for which conditions)

\[ (\mathbf{f}) \mathbf{z}^2 = \mathbf{r} \]

The averaged system
The averaged system

\[ \dot{T} = \mathbf{F}(T) \]

\[ \mathbf{F}(T) = \mathbf{F} + \mathbf{F}_1(T) + \mathbf{F}_2(T) + \cdots \]

where \( \mathbf{F} \) is the effective Hamiltonian in the averaged system.

Finite-mass case

For the finite-mass case, we have

\[ \lim_{t \to \infty} \| \mathbf{F}(T) - \mathbf{F}(t) \| = 0 \]

Next, we consider the general case.

\[ \mathbf{F}(T) = \mathbf{F}(T) + \mathbf{F}_1(T) + \mathbf{F}_2(T) + \cdots \]

The averaged Hamiltonian is given by

\[ \mathbf{F}(T) = \sum_{i=1}^{\infty} \mathbf{F}_i(T) \]

where \( \mathbf{F}_i(T) \) is the effective Hamiltonian in the \( i \)-th perturbation step.

Right: General case

\[ \mathbf{F}(T) = \mathbf{F}(T) + \mathbf{F}_1(T) + \mathbf{F}_2(T) + \cdots \]

We want to know if (and for which conditions)\

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\[ \mathbf{F}(T) = \mathbf{F}(T) + \mathbf{F}_1(T) + \mathbf{F}_2(T) + \cdots \]

Left: General case

\[ \mathbf{F}(T) = \mathbf{F}(T) + \mathbf{F}_1(T) + \mathbf{F}_2(T) + \cdots \]

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The averaged system

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\[ \mathbf{F}(T) = \sum_{i=1}^{\infty} \mathbf{F}_i(T) \]

where \( \mathbf{F}_i(T) \) is the effective Hamiltonian in the \( i \)-th perturbation step.
Step by step

The averaging principle

One perturbation step

Preliminary results

Perturbation construction

Nekhoroshev theory

Step by step

Nearly integrable Hamiltonian systems

Integrable Hamiltonian systems with 2 or more DOF

A starting point: one degree of freedom systems

Integrable and quasi-integrable Hamiltonian systems
We use the time $t$ flow of a "small" Hamiltonian $\epsilon \chi_n$:

\[
(I', \phi') = (\phi, I) + \epsilon \chi_n(I, \phi)
\]

We look for $W$ such that $\tilde{H} = H + W$ is "more integrable" than $H$:

\[
\begin{align*}
(\phi', I') & = (\phi, I) + (\phi', I) + (I', \phi) \\
& \approx \Phi_t(I, \phi) + \epsilon \chi(I, \phi) + \epsilon f(I, \phi)
\end{align*}
\]

Near the identity canonical transformations

We look for $W$ such that $\tilde{H} = H + W$ is "more integrable" than $H$:

\[
\begin{align*}
(\phi', I') & = (\phi, I) + (\phi', I) + (I', \phi) \\
& \approx \Phi_t(I, \phi) + \epsilon \chi(I, \phi) + \epsilon f(I, \phi)
\end{align*}
\]
We use the time $t$ flow of a "small" Hamiltonian $\mathcal{H}$:

$$\left(\omega, \phi \right) = H \circ W$$

We look for $\mathcal{H}$ such that $\mathcal{H} = \phi \circ \mathcal{H} \circ \phi$.
Problem

We use the time flow of a "small" Hamiltonian $\epsilon x$: $(\phi^\tau I', \phi)\Phi = (\phi I)$

How do we generate such transformations?

Consequence (trivial estimate):

$((\phi^\tau I'), I)\epsilon^2 + ((\phi^\tau I'), \phi)\epsilon + ((\phi^\tau I'), H)$

We look for such that $H$ is "more integrable" than $H$, and $(\epsilon^2 (\phi^\tau I'), M) = ((\phi^\tau I'), M)$.
We look for \( \mathcal{W} \) such that \( H' = H \circ \mathcal{W} \). We find \( \mathcal{W} \) as the solution of the equations:

\[
\begin{aligned}
\phi(t, I', \phi') &= \Phi(t, I', \phi') + \epsilon W_t(t, I', \phi', \epsilon) \\
\frac{d}{dt} W_t(t, I', \phi', \epsilon) &= 0
\end{aligned}
\]

for \( t \in [0, T] \).
The averaging principle

One perturbation with finitely many components

Near the identity canonical transformations

\[(\phi^t,\lambda)^\gamma \Phi = (\phi^t,\lambda)\]

We use the time $t$ flow of a "small" Hamiltonian $\varepsilon$:

How do we generate such transformations?

Consequence (trivial estimate):

\[(\phi^t,\lambda)\mathcal{M} + (\phi^t,\lambda)\mathcal{D} + (\phi^t,\lambda)\mathcal{H} = (\phi^t,\lambda)\mathcal{H}\]

We look for such initial Hamiltonian $\mathcal{H}$, and if the Hamiltonian $\mathcal{H}$ is more integrable than $\mathcal{H}$:

\[(\phi^t,\lambda)\mathcal{M} + (\phi^t,\lambda) = (\phi^t,\lambda)\mathcal{H}\]
The averaging principle

Perurbation theory

$F \circ \Phi \epsilon \chi = F + \epsilon R[\chi, F]$

$F \circ \Phi \epsilon \chi = F + \epsilon L \chi F + \epsilon f Rf[\chi, F]$

$L \chi \cdot = \{ \cdot, \chi \}$

$H' = h + \epsilon (f + \{h, \chi \}) + \epsilon f Rf[\chi, h]$

$H'_{\text{int}} = $ integrable and quasi-integrable Hamiltonian systems

$Nekhoroshev theory

Introduction to Hamiltonian perturbation theory

 ether of Hamiltonian perturbation theory

\begin{align*}
\{ F, \chi \} &= \frac{\partial F}{\partial \chi} \\
\{ h, \chi \} &= \frac{\partial h}{\partial \chi}
\end{align*}
\[ 0 = \nu \cdot \Theta \]

A vector \( \nu \in \mathbb{Z}^d \) is called resonant if there exists an integer vector \( \nu \in \mathbb{Z}^d \setminus \{0\} \) such that \( \omega \cdot \nu = d \).

The denominators can be arbitrarily small:

\[
\sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} n_{\nu} \left( \Theta \right) \nu = (\Theta \cdot \chi)
\]

Equation (5) is then solved by

\[
\sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} n_{\nu} \left( \Theta \right) \nu = I
\]
Small divisors and resonances

The denominators can be arbitrarily small:

\[
\sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} \phi(\nu) \approx 0 \implies \sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} \phi(\nu) = 0
\]

Equation (6) is then solved by

\[
\chi = \sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} \phi(\nu)
\]

A vector \( \nu \in \mathbb{Z}^d \setminus \{0\} \) such that

\[
\nu \cdot \omega = d
\]

exists an integer

\[
\nu \in \mathbb{Z}^d \setminus \{0\}
\]

is called resonant.
The averaging principle

Perturbation with finitely many components

Definition (Resonant Lattice)

\[ \mathcal{L} = \{ \nu = d \in \mathbb{Z}^n : \omega \cdot \nu = 0 \} \]

For any \( \omega \in \mathbb{R}^n \) we can define the resonant lattice or resonant modulus of \( \omega \)

Definition (Multiplicity of a Resonance)

If there are \( p \) but not \( p + 1 \) independent vectors \( \nu^{(1)}, \ldots, \nu^{(p)} \in \mathbb{Z}^n \) such that (6) is satisfied, then \( d \) is called the multiplicity of a resonance.
flntegrable and quasiaintegrable fiamiltonian systems

\[ p = J \cap \mathbb{Z} \cap \mathbb{Z}^n \]

The averaging principle

Perturbation with finitely many components

Resonant lattices

Resonant lattice theory

The Hamiltonian perturbation theory

Definition (Resonant lattice)

\[ \left\{ 0 = \alpha \cdot \omega \in \mathbb{Z}^n : \nu \in \mathbb{Z}^n \right\} = J \]

Definition (Resonant lattice or resonant modulus of \( \omega \))

For any \( \mu \in \mathbb{R}^n \) we can define the resonant lattice or resonant modulus of a resonance \( \nu = d \)

Definition (Multiplicity of a Resonance)

If there are \( p \) but not \( p + 1 \) independent vectors \( \nu \) such that (6) is satisfied, then \( p \) is called the multiplicity of a resonance.
Resonant manifold

Definition (_resonant manifold)

\[ \mathcal{J} \in \mathbb{N} \setminus \{0\} = \mathcal{M} \cdot (I) : \mathcal{B} \ni I \} = \mathcal{M} \]

If \( \mathcal{J} \) is any \( d \)-dimensional lattice whose basis is such that

\[ p \geq k \geq 1, \quad N < N, \quad N \geq |(\mathcal{J})|^2 \]

associated to \( \mathcal{J} \) and if \( h \) is non-degenerate, we can define the resonant manifold

\[ \{ I \in \mathcal{B} : \omega(I) \cdot \nu = d \forall \nu \in \mathcal{L} \} \]
Then a function $\chi$ solving (5) in $B^0$ does not exist.

Frobenius series

Thus, choosing $f$ such that $f(I) \neq 0$, $f(I)$ has essentially full rank is non-degenerate.

$B \supset B^0 \ni I \neq 0 \neq \left( \frac{\partial f}{\partial I} \right)_{I=B^0}$

Assume $\forall \nu \in \mathbb{Z}^n \exists \nu' \parallel \nu$ such that $f_{\nu'}(I) \in B^0$.

Then a function $\chi$ solving (5) in $B$ does not exist.

The Poincaré difficulty

The Poincaré hypothesis

Assume that the Hamiltonian perturbation theory

The Hamiltonian perturbation theory
Step by step

1. Introduction to Hamiltonian Perturbation Theory
2. Preliminaries and Technical Tools
3. Nekhoroshev Theory
4. Perturbation with Initially Many Fourier Components
5. One Perturbation Step
6. The Averaging Principle
7. Hamiltonian Perturbation Theory
8. Nearly Integrable Hamiltonian Systems
9. Integrable Hamiltonian Systems with 2 or More DOF
10. A Sliding Point: One Degree of Freedom Systems
11. Integrable and Quasi-Integrable Hamiltonian Systems

Stability Results

Perturbation Construction

Nekhoroshev Theory
We consider the problem of nonresonant case.

Let us suppose that

\[ 0 < N \quad \text{and} \quad \theta(y)^\frac{N}{2}|a| > 1. \]
The averaging principle

One perturbation step with finitely many zjier components

\[ \| \mathbf{f}(\mathbf{I}''_{d}) \| \leq \text{const} \epsilon \]

For \( \epsilon \ll \| \mathbf{a} \| \)

The initial datum is such that

\[ \| \mathbf{a} \| \leq \frac{1}{\sqrt{\epsilon}} \]

where

\[ 0 < N \quad \mathcal{F}_{j} \left( y \right) \]

\( \mathcal{F}_{j} \left( y \right) \)

Let us suppose that

**Non Resonant Case**
Non Resonant Case

Let us suppose that

\[ 0 < N \quad \text{and} \quad \| \mathbf{a} \| > 0 \]

where

\[ \| \mathbf{a} \| \lesssim | \mathbf{a} \cdot \mathbf{a} | \]

The new Hamiltonian will be

\[ H' (I', \mathbf{\phi}') = h (I') + \varepsilon f (I') + \varepsilon f' (I', \mathbf{\phi}') \]

The initial datum is such that

\[ \| \mathbf{a} \| \lesssim | \mathbf{a} \cdot \mathbf{a} | \]

Conclusion: for any finite component

\[ \| (0)_{I'} - (I)_{I'} \| \leq \text{const} \varepsilon \]

\[ (\mathbf{a}, I)_{I'} d I^2 + (I') d I^2 + (J) d \mathbf{a} = (\mathbf{a}, I') H \]

\[ \mathbf{f}' = [\chi, \mathbf{a}] + \mathbf{R} [\chi, h] = \{ \mathbf{f}, \chi \} - e \{ (\mathbf{f} - \mathbf{f}) , \chi \} + O (\varepsilon) \]

\[ \| I' (t) - I' (d) \| \leq \text{const} \varepsilon \]

for \( |t| \leq \varepsilon / \varepsilon \)
Consideration: \((\|\mathbf{v}_{d} - \mathbf{v}_{0}\|)\) for \(\varepsilon \sim \max |\mathbf{a}|\). The initial datum is such that

\[ |\mathbf{a} \cdot \mathbf{v}| \gg \sqrt{\varepsilon} \text{ for } d < |\mathbf{a}| \leq N. \]

The new Hamiltonian will be

\[ H' = H(\mathbf{I}', \phi') = h(\mathbf{I}') + \varepsilon_{f}(\mathbf{I}') + \varepsilon_{f}(\mathbf{I}', \phi'). \]

The nonresonant case

Let us suppose that

\[ 0 < N \quad \sigma_{a} \theta(\xi, \mathbf{v}) \quad \text{for} \quad \frac{N}{\varepsilon} |\mathbf{a}| > 0. \]
The initial datum is taken near to $M$ but far from other resonances.
The initial datum is taken near to the Nekhoroshev theory.

Resonant case

The new Hamiltonian will be

\[ H' = (I', \phi') + \varepsilon g(I', \phi') + \varepsilon f(I', \phi') + \sum_{\nu \in L} \lambda_{\nu}(I')e^{i\nu \cdot \phi'} \]

where

\[ \lambda_{\nu}(I') = \int_0^1 \phi'(t, I', \phi') dt \]

The remainder is taken near to another Hamiltonian.
The averaging principle

One perturbation step

Perturbation with finitely many Fourier components

The new Hamiltonian will be normal form.

The resonances are the initial datum is taken near to $\mathcal{M}$ but far from other resonances.

Hamiltonian $
\hat{H}\left(I', \phi'\right) = H(I') + \varepsilon g(I', \phi') + \varepsilon f f' (I', \phi')
$

Normal form:

$g(I', \phi') = \sum_{\nu \in \mathcal{L}} f_{\nu} (I') e^{i \nu \cdot \phi'}$

Remainder:

$\delta z = g(I', \phi')$

Notice:

$\dot{I}' = \varepsilon \sum_{\nu \in \mathcal{L}} i_{\nu} f_{\nu} (I') e^{i \nu \cdot \phi'}$

Introduction to Hamiltonian perturbation theory

Familtonian perturbation theory

Nekhoroshev theory
The initial datum is taken near to $\mathcal{M}$ but far from other resonances.

The new Hamiltonian will be

$$
\mathcal{H}'(I', \phi') = \mathcal{H}(I', \phi') + \epsilon \sum_{\nu \in \mathcal{L}} f_{\nu}(I', \phi') e^{i \nu \cdot \phi'}
$$

Conclude:

$$
\mathcal{H} = \mathcal{H}_{\text{rem}} + \chi \epsilon\|\mathcal{H}' - \mathcal{H}_{\text{rem}}\|
$$
The plane of fast drift

Normal form: produces a transversal to \( \Pi(I_0) \)

Remained: produces motions of fast drift

\( \Pi(I_0) \parallel L \), called plane

Problem

Bound motions on \( \Pi(I_0) \)

Introduction to Hamiltonian perturbation theory

Hamiltonian perturbation theory

Nekhoroshev theory

Integrable and quasintegrable Hamiltonian systems
**Problem**

- Transversal to $\Pi_0$: produces motions of fast drift.
- Remainder: produces motions parallel to $\Pi_0$, called plane dynamics, flat on the plane $\Pi_0$.
- Normal form: produces a plane of fast drift.

**The plane of fast drift**

- Transversal to $\Pi_0$: many possible components.
Step by step

Stability results
Perturbation construction
Preliminary and technical tools
Nekhoroshev theory

Perturbation with finitely many Fourier components
One perturbation step
The averaging principle
Hamiltonian perturbation theory

Nearly integrable Hamiltonian systems
Integrable Hamiltonian systems with 2 or more DOF
A single point: one degree of freedom systems
Integrable and quasi-integrable Hamiltonian systems
Stability of the actions also in the resonant case

What we want:

- Some assumptions which make us able to deduce the stability of the actions also in the resonant case
- Each perturbation step
- To estimate the transformation and the new functions at each
- To study the geography of resonances
- That ε is small enough
- That Ω is analytic (in a suitable complex extended domain)

What we need:

- To iterate the perturbation step enough times
- J with full Fourier series

Plan of work

- Introduction to Hamiltonian Perturbation Theory
introduction to fiamiltonian perturbation theory

Vb ¨arini flntroduction to fiamiltonian perturbation theory

stability of the actions also in the resonant case

some assumptions which make us able to deduce the

each perturbation step

is to estimate the transformation and the new functions at

that ε is small enough

that H is analytic in a suitable complex extended domain

What we need:

is to relate the perturbation step enough times

f with full Fourier series

What we want:

plan of work
The function \( h \) is \( m \)-quasi-convex if \( m \) is such that at least one of the inequalities

\[
\omega(I) \cdot \xi > m \xi, \quad h''(I) \xi \cdot \xi \geq m \xi^2
\]

holds for each \( \xi \in \mathbb{R}^d \). More generally, \( h \) is said to be \( (m, \xi) \)-quasi-convex if

\[
\exists l > 0 \text{ such that at every point } I \in B \text{ at least one of the inequalities }
\omega(I) \cdot \xi > l \xi, \quad h''(I) \xi \cdot \xi \geq m \xi^2
\]

holds for each \( \xi \in \mathbb{R}^d \).
The role of quasiconvexity
The role of quasi-convexity

Smaller remainder $\Rightarrow$ longer confinement

between two nearby level surfaces and is
the motion of the actions is confined
$\leftrightarrow$ oscillation bounded by
Energy conservation provides confinement:
the form of confining ellipsoids around $I$

$0 = A \cdot (\gamma)^{m} \Rightarrow \nabla \cdot A \cdot (\gamma)^{m} + \gamma^{m+1} = (A + \gamma^{m}) \gamma^{m}$

$\bigcup_{\nu} (\gamma)^{m} = \gamma$

and $\gamma$ are transversal and

If $I$ is quasi-convex in $B$, then
The role of quasi-convexity

...
The role of quasi-convexity

Energy conservation provides confinement: the motion of the actions is confined between two nearby level surfaces and is bounded by $\varepsilon = \frac{1}{\sqrt{2}} + B^{\varepsilon}$.

The form of confining ellipsoids around $I^*$ restricted to $U(\varepsilon)$ has level surfaces in $\mathcal{L}(I)$ and are transversal and $\varepsilon$-smooth. If $I^*$ is quasi-convex in $\mathcal{L}$, then

\[ 0 = \lambda \cdot (\varepsilon) \varepsilon \iff \varepsilon \in \mathcal{L} \]

\[ \cdots + \lambda \cdot (\varepsilon) \varepsilon + \mathcal{L}(I) = (\mathcal{L} + (\varepsilon) \varepsilon) \mathcal{L} \]

Smaller remainder $\Rightarrow$ longer confinement

$\varepsilon$-smooth remainder $\Rightarrow$ longer confinement

$V_{\varepsilon}$ barrier introduction to Hamiltonian perturbation theory

Hamiltonian perturbation theory

Hamiltonian perturbation theory

Problems and exercises in Hamiltonian perturbation theory

Problems and exercises in Hamiltonian perturbation theory

Problems and exercises in Hamiltonian perturbation theory

Problems and exercises in Hamiltonian perturbation theory
Smaller remainder $\Rightarrow$ longer confinement

The motion of the actions is confined $\iff$ $\varepsilon \in \mathfrak{L}$

Energy conservation provides confinement:
the form of concentric ellipsoids around $\mathcal{I}$

If restricted to $\mathcal{U}_C(\mathcal{I})$ has level surfaces in $\mathfrak{L}$

$0 = \mathfrak{A} \cdot (\mathcal{I})$ $\iff$ $\mathfrak{L} \in \mathfrak{A}$

$\cdots + \mathfrak{A} \cdot \mathfrak{A} (\mathcal{I}) \mathfrak{U}_C^2 + \mathfrak{A} \cdot (\mathcal{I}) \mathfrak{U} = (\mathfrak{A} + \mathfrak{A}) \mathfrak{U}$

$\mathfrak{W} \bigcup \mathfrak{U} (\mathcal{I}) \mathfrak{U} = \mathcal{I}$

$\mathfrak{U}$ and $\mathfrak{W}$ are transversal and $\mathfrak{W} \bigcup \mathfrak{U} (\mathcal{I}) \mathfrak{U}$

If $\mathfrak{L}$ is quasiconvex in $\mathfrak{E}$, then

The role of quasiconvexity
The role of quasistatic convexity
The role of quasiconvexity
Unary motions occur precisely along these lines

\[ \omega(I) = (I_e, -I_f) \]

\[ h'' = (e_d) - (e) \]

Lack of quasi-convexity on the lines \( h \):

\[ \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \mu, \quad (\varepsilon - 1) = (I) \]

Now we can understand better the previous example of instability:

Counterexample
We assume $\mu$ analytic in $D^\rho$ for some $\rho$.

\[ \{ \mathbf{p}_i \cdots \mathbf{p}_1 = f, \mathbf{d}_i \leq |\mathbf{d}_i|, \mathbf{m}_i \in \mathcal{M} \} = \mathcal{M} \]

\[ \{ \mathbf{b} \in \mathbf{I}, \mathbf{p}_i \cdots \mathbf{p}_1 = f, \mathbf{d}_i \leq |\mathbf{d}_i|, \mathbf{m}_i \in \mathcal{M} \} = \mathcal{M} \]

\[ (\mathbf{d}_i \cdot \mathbf{d}_j) = \mathbf{d}_i \cdot \mathbf{d}_j \]

One works in complex domains of the form...

Domains
One exploits this fact to work only with finitely many Fourier components of the perturbation $V$.

\[ A, \forall \nu \in \mathbb{Z} \quad \| F \|_\infty = \sup_{\nu \in \mathbb{Z}} \| F_\nu \|_\infty, \]

The Fourier norm is especially suited to handle the basic equation (5) for $X$.

The Fourier norm of the function $F$ is defined by

\[ F : \mathbb{D}^d \rightarrow \mathbb{C}, \text{ analytic} \]
Step by step

1. Integrable and quasi-integrable Hamiltonian systems
   - A starting point: one degree of freedom systems
   - Integrable Hamiltonian systems with 2 or more DOF
   - Nearly integrable Hamiltonian systems

2. Hamiltonian perturbation theory
   - The averaging principle
   - One perturbation step
   - Perturbation with finitely many Fourier components

3. Nekhoroshev theory
   - Preliminaries and technical tools
   - Perturbation construction
   - Stability results
The idea is to work with finitely many Fourier components.

Let $f$ be analytic in $D_\rho r$ then

$$\|f\|_{N>\rho} \leq 2^{1/\rho}\|f\|_{N<\rho}$$

**Lemma**

We take advantage of the following lemma:

We decompose $f$ as

$$f = f_{N<\rho} + f_{N>\rho}$$

We introduce a cut-off $N$, it will be $N = N(\epsilon)$. Usually we choose conveniently $N(\epsilon)$.
We introduce a cut-off \( N \). It will be \( N = N(\varepsilon) \), usually \( N \propto \varepsilon \). At the end of the perturbation construction one will choose conveniently \( N(\varepsilon) \).

The idea is to work with finitely many Fourier components.

### Lemma

If \( f \) is analytic in \( D^\rho r \) then

\[
\|f\|_{N<\frac{\varepsilon}{2}} \leq 2^{\rho} \|f\|_{N<\frac{\varepsilon}{2}}
\]

\( \text{if } f \text{ is analytic in } D^\rho r \text{ then} \)

\[
\|f\|_{N<\frac{\varepsilon}{2}} \leq 2^{\rho} \|f\|_{N<\frac{\varepsilon}{2}}
\]

We decompose \( f \) as

\[
N^<_\varepsilon + N^>_\varepsilon = f
\]
The idea is to work with finitely many Fourier components.

We introduce a cut-off $N$. It will be $N = N(\epsilon)$, usually $N \propto \epsilon$.

At the end of the perturbation construction one will choose conveniently $N(\epsilon)$.

We take advantage of the following lemma:

We decompose $f$ as

\[ f = f_{\leq N} + f_{> N}, \]

where $f_{\leq N}$ is the infrared part, $f_{> N}$ is the ultraviolet part,

\[ f_{\leq N} + f_{> N} = f. \]

If $f$ is analytic in $D_{\rho \epsilon}$ then

\[ \| f_{> N} \|_{\rho \epsilon} / \| f_{\leq N} \|_{\rho \epsilon} \leq e^{-\epsilon}. \]
The idea is to work with finitely many Fourier components. We introduce a cut-off $N$. It will be $N = N(\varepsilon)$ usually $N \propto \varepsilon$. At the end of the perturbation construction one will choose conveniently $N(\varepsilon)$.

We decompose $f$ as $f = f_{\leq N} + f_{> N}$.

We take advantage of the following lemma:

**Lemma**

If $f$ is analytic in $D_\rho$, then $\|f\|_{L^2} \leq e^{\varepsilon D_\rho |f|_{L^2}}$. 

$f_{\leq N}$ = infrared part; $f_{> N}$ = ultraviolet part.
We use the Lie method to generate a near the identity canonical transformation.

\[
\begin{align*}
\text{First perturbation step} & \quad \text{Hamiltonian theory} \\
\text{Hamiltonian function} & \quad \text{Hamiltonian perturbation theory} \\
\text{Resonant case} & \quad \text{Nonresonant case}
\end{align*}
\]
The transformed Hamiltonian will be,
\[ \chi \sum_{N \leq j < N} \frac{m \cdot \ell I}{(y)^{\frac{3}{2}}} = (\chi \cdot y) \chi \]

We use the Lie method to generate a near the identity canonical transformation.
The term is inessential (smaller than $N^{-2}$) and the integrable systems.

Non-resonant case: $n = \mathcal{B}$

$\{0\} = \mathcal{J}(\mathcal{B}) \no\{0\} = \mathcal{B}$

The mechanism of confinement still works.

\begin{align*}
W \circ N < I &= (N, N) \mathcal{B} + [N, N \chi] \mathcal{B} = I \quad (N, N = I^2) \mathcal{B} = \mathcal{B} \\
N < I^2 + \mathcal{B}^2 + \mathcal{B}^2 + \mathcal{B} = \mathcal{H}
\end{align*}

The transformed Hamiltonian will be

$$\mathcal{W} \mathcal{B} \mathcal{H} \mathcal{B} \mathcal{W} = (\mathcal{Y} \mathcal{Y}) \mathcal{Y}$$

canonical transformation

We use the Lie method to generate a near the identity

First perturbation step
We use the Lie method to generate a near the identity canonical transformation

\[ \chi(I, \varphi) = \sum_{\nu} \frac{1}{\nu!} \nu \cdot \omega e^{i \nu \cdot \varphi} \]

The transformed Hamiltonian will be

\[ H' = h + \varepsilon g + \varepsilon f' + \varepsilon \tilde{f} > N, g = \prod_{L} f \leq N, f' = \text{Re} [\chi, f \leq N] + Rf [\chi, h], \tilde{f} > N = f > N \circ w \]

The mechanism of confinement still works. The term is inessential (smaller than) and the

\[ M \circ N < I = N < I - [y \cdot \chi]^2 y + [N \geq I \cdot \chi]^2 y = I \quad N \geq I \quad |x| = \mathcal{B} \]

\[ I < I - \varepsilon \mathcal{B}^2 + \varepsilon \mathcal{B}^2 + y = \mathcal{H} \]

Non-resonant case: \( g = g(I') \)

Resonant case: \( n g = g(I', \varphi') \)

First perturbation step
We use the Lie method to generate a near the identity canonical transformation.

\[ \begin{align*}
&\psi \cdot \eta^\theta (y^N) \sum_{N \geq 1} = (\psi \cdot \eta)^\chi \\
\end{align*} \]

First perturbation step

Non-resonant case: \( \{0\} = \mathcal{F}(\eta) \mathcal{B} = \mathcal{B} \)

Resonant case: \( \{0\} = \mathcal{F}(\eta) \mathcal{B} = \mathcal{B} \)

Mechanism of confinement still works.

The term is inessential (smaller than \( \varepsilon \)) and the Hamiltonian will be

\[\begin{align*}
\mathcal{M} \circ N < I & = N < I \\
\mathcal{M} \circ N < I & = N < I \quad [y \cdot \chi]^2 + [N \geq I \cdot \chi] \mathcal{V} & = \mathcal{J} \\
N \geq I^2 & = \mathcal{B} \\
N < I^2 & = \mathcal{B} \\
\end{align*}\]
Heteroclinic points we can reduce the remainder of a constant.

At each step one has to estimate the norm of each function.

Output of the s-th step: as above, with \( s + 1 \) in place of \( s \).

\[
\chi_{\mathbb{M}^p}(\psi)(s) = (\phi \cdot \psi)(s)
\]

\[
\nu = (s) + \mu = (s)H
\]

Input of the (s+1)-th step:

\[
\nu = (s) + \mu = (s)H
\]

If \( \mu \) is small enough the previous procedure can be iterated.
Rectangular at each step: 

\[ \sum_{\nu \in \mathcal{L}} g(s) L(\nu) e_i \nu \cdot \phi \approx (s) H \]

Output of the 0-th step: as above, with \( s + 1 \) in place of \( s \).

Input of the \( s \)-th step:

By induction:

If \( \varepsilon \) is small enough, the previous procedure can be iterated.
Rectangular step: each step we can reduce the remainder of a constant.
Key point: we can reduce the remainder of a constant.
To check that the iteration is possible and moreover that:
At each step one has to estimate the norm of each function.
Output of the s-th step: as above, write $s + 1$ in place of $s$.

\[ \sigma_{\gamma}(l)^{(s)} = (\gamma)_{(s)} \quad \text{(by induction)} \]

If it is small enough, the previous procedure can be iterated.

Iterative procedure
Rect at each step:\n\[ f(s) + s \epsilon^2 f(s) \sim (s + 1) \epsilon s + f(s) \]

Output of the s-th step: as above, with \( s + 1 \) in place of \( s \).

\[ \phi(s) = (\phi(s), \phi(s)) \]

Input of the s-th step: (by induction)

If \( \epsilon s + e \) is small enough the previous procedure can be iterated.
Iterative procedure

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(s)(H_i^{(s)+1} + \epsilon \sum g_i(s) + \epsilon f_i(s+\epsilon)) = (s)(H)</td>
</tr>
</tbody>
</table>

Input of the \(s\)-th step:

- (By induction)

Output of the \(s\)-th step:

- As above, with \(s+1\) in place of \(s\).

At each step one has to estimate the norm of each function

Key point: we can reduce the remainder of a constant

If \(\epsilon\) is small enough the previous procedure can be iterated

Factor at each step: \(\sim (1+\epsilon)^{s+2} \sim (1+\epsilon)^{s+2} \)
After $r$ perturbation steps, $r \sim N$, the transformed Hamiltonian will be

\begin{equation}
H = (\phi^1, I)^2 + (\phi^2, \dot{I}) + \epsilon (\phi^1, I) + \frac{1}{2} \epsilon^2 (\phi^1, \phi^2)
\end{equation}

where $N$ is the number of resonant actions. In the non-resonant case, $\mathcal{O}$ depends only on $I$.
1964

 sellers and \( N \gg 1 \). Arnold diffusion takes place (Arnold).

 Such time scale is the best one can do in the longer time

 long.

 Now the confinement of the actions will be exponentially

 The final remainder is exponentially small as \( N \gg 1 \).

 in the non resonant case \( \beta \) depends only on \( I_1 \).

 \( N \gg 1 \).

 \( (\delta^2 \mathcal{H})^2 \theta^2 + (\delta^2 \mathcal{H})^2 \theta^2 + (\mathcal{H})^4 = (\delta^2 \mathcal{H})^2 \mathcal{H} \)

 After \( \mathcal{H} \) perturbation steps, \( N \gg 1 \). The transformed Hamiltonian

 Normal Form

 Integration by parts

 Hamiltonian

 Invariant

 Transformations

 Hamiltonian perturbation theory

 Introduction to Hamiltonian perturbation theory

 Preliminary and technical tools

 Perturbation construction

 Symplectic reduction and Moser's theorems
The normal form is adapted to the resonance we are considering: for each $\mathcal{L}$ we must produce a canonical transformation $\mathcal{W}$ such that $\mathcal{H} \circ \mathcal{W}$ is as in (14). The total oscillation of the actions is estimated by

$\Delta I \leq \Delta I' + f \Delta w$, $\Delta I = \| I(t) - I(d) \|$, $\Delta w = \| I' - I \|$.

Resonant regions, resonant manifolds and their geography depend on $N(t, \varepsilon)$. Where $\omega_L$ is the orthogonal projection of $\omega$ onto $\mathcal{L}$,

$\mathcal{W}_{\text{resonant manifolds}}$ and their geography depend on $N_s \varepsilon^2 T_b$. To construct each normal form one works in a resonant region.

\begin{align*}
\| I - I \| &= \| I(0) - I(d) \| = I \| + 2 \| \mathcal{W} \| I, \mathcal{W} \geq I \| \V_{\text{barini introduction to hamiltonian perturbation theory}}
\end{align*}
The normal form is adapted to the resonances we are considering: for each $\mathcal{L}$ we must produce a canonical transformation $\tilde{\mathcal{L}}$ such that $\tilde{\mathcal{L}} \circ \mathcal{L}$ is as in (1.1). To construct each normal form one works in a "resonant region" and then produces a canonical transformation $\tilde{\mathcal{L}}$ of it. The total oscillation of the actions is estimated by

$$\Delta I \leq \Delta I' + f \Delta w,$$

where $\Delta I' = \| I(t) - I(d) \|$ and $\Delta w = \| I' - I \|$. The integral over the $P\gamma$ of the expression $(1) \gamma \geq \| (y) \gamma \|$ gives $\mathcal{R}$. The resonant regions, resonant manifolds and their geography depend on $N(1.0.)$. Resonant integrals and quasiaintegrable Hamiltonian systems...
The normal form is adapted to the resonance we are considering. For each $L$ we must produce a canonical transformation $W$ such that $\mathcal{H} \circ W$ is as in (1.1). The total oscillation of the actions is estimated by

\[ \Delta I \leq \Delta I' + f \Delta w, \]

\[ \Delta I = \| I(t) - I(d) \|, \quad \Delta w = \| I' - I \|. \]
The normal form is adapted to the resonances we are considering for each $\mathcal{L}$ we must produce a canonical transformation $\mathcal{W}$ such that $H \circ \mathcal{W}$ is as in (11).

To construct each normal form one works in a resonant region with $\mathcal{W}$ such that $H \circ \mathcal{W}$ is as in (11).
Step by step
Assume that the Hamiltonian \( H = h + \epsilon f \) is analytic in \( D_\rho \) and \( h \) is \( l, m \)-quasiconvex in \( B \). Then for any motion \( (I(t), \phi(t)) \) with initial datum \( (I_0, \phi_0) \in B \times \mathbb{R}^n \), satisfying the estimate \( \|I(t) - I_0\| \leq C_\epsilon \), for \( |t| \leq T(\epsilon) \), where \( C_\epsilon \) is \( \epsilon \)-analytic in \( D_\rho \) and \( h \).

Possible choices of the exponents are:

\[ a = a(T_\rho \Theta) \text{ for } t \geq |I| \]

Theorem (Nekhoroshev) (1992), Poschel (1993), Lochak & Neishtadt

Stability estimates
The methods of perturbation theory can be adapted to degenerate systems, basically as if the degenerate variables \((b, d)\) would not exist. The theorem gives stability only for the nondegenerate action variables nothing is known, already on the time-scale \(\varepsilon^2\).

\[
\frac{d}{dt} (b, d) + (\phi, \psi) = (b, d) \cdot (\phi, \psi)
\]

\[
H = (b, d) \cdot (\phi, \psi) + \varepsilon (\phi, \psi)
\]
The methods of perturbation theory can be adapted to degenerate variables \((b, d)\) as well as non-degenerate variables \((\phi, \psi)\).

\[
\begin{align*}
(b, d) & \rightarrow (b, d, \phi, \psi) \\
& \rightarrow (b, d, \phi, \psi)^T + (\phi, \psi)^T = (b, d, \phi, \psi)^T + (\phi, \psi)^T
\end{align*}
\]
Degenerate systems

\( H(I, \varphi, p, q) = h(I) + \epsilon f(I, \varphi, p, q) \)

...
Degenerate systems

The methods of perturbation theory can be adapted to non-degenerate variables

\((\mathbf{b} \cdot \mathbf{d})\) on actions: nothing is known, already on the time-scale \(t/\varepsilon\),

The theorem gives stability only for the non-degenerate

If the quasi-converse Nekhoroshev theorem can still be

viable, \((\mathbf{b} \cdot \mathbf{d})\) would not exist.

degenerate systems. Basically all the degenerate

\([a, b, c, d]\) = \((\mathbf{b} \cdot \mathbf{d})\)
degenerate variables

\((\mathbf{b} \cdot \mathbf{d} \cdot \phi, \mathbf{i})\) + \(|\mathbf{i}| = (\mathbf{b} \cdot \mathbf{d} \cdot \phi, \mathbf{i}) H\)
Informations about the stability of the degenerate actions.

For degenerate systems an ad hoc work, adapted to the specific problem, is required to obtain, when possible.

If chaotic motions may occur:

- Everything could happen in a time $t/\varepsilon$.
- In lack of further informations on the perturbation, the degenerate variables $= (b \cdot d)$
  non degenerate variables $= (\phi \cdot j)$

$$H = (b \cdot d \cdot \phi \cdot j) + (j) \cdot (b \cdot d \cdot \phi \cdot j)$$

**Degenerate systems**

- Specific cases
- Hamiltonian configuration
- Hamiltonian perturbation theory
- Degenerate and non degenerate Hamiltonian systems
Informations about the stability of the degenerate actions.

For degenerate system an ad hoc work, adapted to the

- Chaotic motions may occur.
- Everything could happen in a time $t/\varepsilon$.

In lack of further informations on the perturbation,

\[ (\mathbf{b} \cdot \mathbf{d}, \phi, \gamma, \varepsilon) + (\mathbf{j}) = (\mathbf{b} \cdot \mathbf{d}, \phi, \gamma, \varepsilon) H \]

Degenerate systems
Degenerate systems

\[ H(\mathbf{I}, \mathbf{\phi}, \mathbf{p}, \mathbf{q}) = h(\mathbf{I}) + \varepsilon f(\mathbf{I}, \mathbf{\phi}, \mathbf{p}, \mathbf{q}) \]

- (\mathbf{\phi}, \mathbf{p}, \mathbf{q}) = \text{non degenerate variables}
- (\mathbf{p}, \mathbf{q}) = \text{degenerate variables}
- In lack of further informations on the perturbation, everything could happen in a time \(1/\varepsilon\).
- Chaotic motions may occur.

For degenerate system an ad hoc work, adapted to the specific problem, is required to obtain, when possible, informations about the stability of the degenerate actions.
Introduction to Hamiltonian Perturbation Theory

Preliminaries and Technical Tools

Nekhoroshev Theory

Perturbation Construction

Stability Results

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Integrable and Quasintegrable Hamiltonian Systems

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The End