Ultimate Polynomial Time

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Abstract

The class $\mathcal{UP}$ of ‘ultimate polynomial time’ problems over $\mathbb{C}$ is introduced; it contains the class $\mathcal{P}$ of polynomial time problems over $\mathbb{C}$.

The $\tau$-Conjecture for polynomials implies that $\mathcal{UP}$ does not contain the class of non-deterministic polynomial time problems definable without constants over $\mathbb{C}$. This latest statement implies that $\mathcal{P} \neq \mathcal{NP}$ over $\mathbb{C}$.

A notion of ‘ultimate complexity’ of a problem is suggested. It provides lower bounds for the complexity of structured problems.

1 Introduction

A model of Computation and Complexity over a ring was developed in [?] and [?], generalizing the classical $\mathcal{NP}$-completeness theory [?]. Of particular interest is the model of Complexity over the ring $\mathbb{C}$ of complex numbers.

In the model of complexity over $\mathbb{C}$, a machine is allowed to input, to output and to store complex numbers, to compute polynomials and to branch on equality (See the textbook [?] for background). This model shares some of the features of the classical (Turing) model of computation (There is a discussion in [?]). It is known [? , ?] that the hypothesis $\mathcal{BPP} \nsubseteq \mathcal{NP}$ in the Turing setting implies $\mathcal{P} \neq \mathcal{NP}$ over $\mathbb{C}$. ($\mathcal{BPP}$ stands for Bounded

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Probability Polynomial Time. If $BPP$ would happen to contain $NP$, then there would be polynomial time randomized algorithms for such tasks as factorizing large integers or breaking most modern cryptographic systems).

In $[?, ?, ?]$, the hypothesis $P \neq NP$ over the Complex numbers was related to a number-theoretical conjecture. Define a straight-line program as a list

$$s_0 = 1, \ s_1 = x, \ s_2, \ldots, \ s_\tau$$

where $s_i$ is, for $i \geq 2$, either $s_j + s_k$, $s_j - s_k$ or $s_js_k$, for some $j, k < i$. Each $s_i$ is thus a polynomial in $x$. The straight-line program is said to compute the polynomial $s_\tau(x)$.

Given a polynomial $f \in \mathbb{Z}[x]$, the quantity $\tau(f)$ is defined as the smallest $\tau$ such that there exists a straight-line program $s_0, \ldots, s_\tau$ computing $f(x)$. For instance, $\tau(x^{2n} - 1) = 2 + n$. Similarly, if $g \in \mathbb{Z}[x_1, \ldots, x_n]$, then $\tau(g)$ is the minimal length of a straight-line program $s_0 = 1, s_1 = x_1, \ldots, s_n = x_n, s_{n+1}, \ldots, s_\tau = g(x)$.

The $\tau$ Conjecture for Polynomials. There is a constant $a > 0$ such that for any univariate polynomial $f \in \mathbb{Z}[x]$,

$$n(f) < \tau(f)^a$$

where $n(f)$ is the number of integer zeros of $f$, without multiplicity.

It is known $[?]$ that the $\tau$-Conjecture for polynomials implies $P \neq NP$ over $\mathbb{C}$. A main step towards this result is the fact that, if the $\tau$-Conjecture is true, then the polynomials

$$p_d(x) = (x - 1)(x - 2)\cdots(x - d)$$

are ultimately hard to compute. This means that there cannot be constants $a$ and $b$ such that, for any degree $d$, for some non-zero polynomial $f$ (depending on $d$), we would have

$$\tau ( p_d(x) f(x) ) < a (\log_2 d)^b$$

Therefore, all non-zero multiples of $p_d$ are hard to compute, hence the wording ultimately hard.
The goal of this paper is to define a new complexity class $\mathcal{UP}$, of *ultimate polynomial time* problems. This class will contain $\mathcal{P} \cap \mathcal{K}$, where $\mathcal{P}$ is the class of problems decidable in polynomial time and $\mathcal{K}$ is the class of problems definable without constants (See [?] and Definition 1 below). Moreover:

**Theorem 1.** The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are true:

(a) The $\tau$-conjecture for polynomials.

(b) $\forall d, p_d$ is ultimately hard to compute.

(c) $\mathcal{UP} \not\supseteq \mathcal{NP} \cap \mathcal{K}$ over $\mathbb{C}$.

(d) $\mathcal{P} \neq \mathcal{NP}$ over $\mathbb{C}$.

The implication (a) $\Rightarrow$ (b) $\Rightarrow$ (d) appears in [?], the hypothesis (c) in-between is new. It is at least as likely as the $\tau$-conjecture, while still implying $\mathcal{P} \neq \mathcal{NP}$.

We will also show a $\mathcal{NP}$-hardness result for the class $\mathcal{UP}$: there is a structured problem $(HN, HN^{yes}) \in \mathcal{NP} \cap \mathcal{K}$, such that:

**Theorem 2.** $\mathcal{UP} \not\supseteq \mathcal{NP} \cap \mathcal{K}$ over $\mathbb{C}$ if and only if $(HN, HN^{yes}) \not\in \mathcal{UP}$ over $\mathbb{C}$.

The problem $(HN, HN^{yes})$ is precisely the (structured) Hilbert Nullstellensatz, known to be $\mathcal{NP}$-complete over $\mathbb{C}$ ([?]).

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## 2 Background and Notations

Recall from [?] that $\mathbb{C}^\infty$ is the disjoint union

$\mathbb{C}^\infty = \bigsqcup_{i=0,1,...} \mathbb{C}^i$ 

This means that there is a well-defined size function,

$\text{Size} : \mathbb{C}^\infty \rightarrow \mathbb{N}$

$x \mapsto \text{Size}(x) = i$ such that $x \in \mathbb{C}^i$
A decision problem \(X\) is a subset of \(\mathbb{C}^\infty\). It is in the class \(P\) if and only if there is a machine \(M\) over \(\mathbb{C}\), that terminates for any input \(x\) in time bounded by a polynomial on \(\text{Size}(x)\), and such that

\[
M(x) = 0 \iff x \in X
\]

where \(M(x)\) is the result of running \(M\) with input \(x\). Without loss of generality we may assume that \(M(x) \in \{0; 1\}\).

Under some circumstances, it is possible to assume that the machine \(M\) above has only coefficients 0 or 1 (This is called a constant-free machine). However, one may have to replace problem \(X\) over \(\mathbb{C}\) by problem \(X \cap \mathbb{Z}\) over \(\mathbb{Z}\), with unit cost. (This is the contents of Propositions 3 and 9 of Chapter 7 of \([?]\)). In order to avoid this technical complication and keep the same problem over \(\mathbb{C}\), we will follow another approach to Elimination of Constants.

This approach was introduced by Koiran in \([?]\). The idea is to consider only machines for a subclass of problems. This subclass will contain most of the interesting examples, while precluding pathological cases such as \(X = \{\pi\}\).

**Definition 1 (Koiran).** A problem \(L\) is said to be definable without constants if for each input size \(n\) there is a formula \(F_n\) in the first order theory of \(\mathbb{C}\) such that 0 and 1 are the only constants occurring in \(F_n\), and for any \(x \in \mathbb{C}^n\), \(x \in L\) if and only if \(F_n(x)\) is true (there is no restriction on the size of \(F_n\).

For future reference, we quote below Theorem 2 of \([?]\). The original statements of both Definition 1 and Theorem 3 are actually more general (for any algebraically closed field of characteristic 0).

**Theorem 3 (Koiran).** Let \(L \subseteq K^\infty\) be a problem which is definable without constants. If \(L \in P\), \(L\) can be recognized in polynomial time by a constant-free machine.

The class of all the problems definable without constants will be denoted by \(K\).

We will need crucially in the sequel the notion of a structured problem. A structured problem is a pair \((X, X^{\text{yes}})\), \(X^{\text{yes}} \subseteq X \subseteq \mathbb{C}^\infty\). A non-structured problem \(X\) can always be written as the structured problem \((\mathbb{C}^\infty, X)\). The class \(\mathcal{UP}\) will be meaningful only as a class of structured problems. But first of all, recall that
Definition 2. A structured problem $(X, X^{yes})$ belongs to the class $\mathcal{P}$ if and only if $X \in \mathcal{P}$ and $X^{yes} \in \mathcal{P}$.

Definition 3. A structured problem $(X, X^{yes})$ belongs to the class $\mathcal{K}$ if and only if $X \in \mathcal{K}$ and $X^{yes} \in \mathcal{K}$.

Definition 4. A structured problem $(X, X^{yes})$ belongs to the class $\mathcal{NP}$ if and only if:

1. The problem $X$ belongs to the class $\mathcal{P}$.
2. There is a machine $M$ with input $x, g$ such that $x \in X$ and $\exists g \in \mathbb{C}^\infty$ s.t. $M(x, g) = 0 \iff x \in X^{yes}$
3. Furthermore, there is a polynomial $p$ such that, for all $x \in X^{yes}$, there is $g \in \mathbb{C}^\infty$ such that $M(x, g) = 0$ and the running time of $M$ with input $x, g$ is no more than $p(\text{Size}(x))$.

Example 1. Let $HN$ be the class of all lists $(m, n, f_1, \cdots, f_m)$ where $f_1, \cdots, f_m$ are polynomials in $n$ variables. Each polynomial $f = \sum f_I x^I$ is represented sparsely by a list of monomials $(S, m_1, \cdots, m_S)$, where each monomial is a list $(f_I, I_1, \cdots, I_n)$.

An important convention to have in mind: integers appearing in the definition of a problem should be represented in bit representation. In this case, $m, n, S, I_j$ are all lists of zeros and ones. Complex values are represented by one complex number. With this convention, $HN$ is clearly in the class $\mathcal{P}$.

We also define $HN^{yes}$ as the subset of polynomial systems in $HN$ that have a common root over $\mathbb{C}$. The definition above of the structured problem $(HN, HN^{yes})$ can be translated into first order constant-free formulae over $\mathbb{C}$. Therefore, $(HN, HN^{yes}) \in \mathcal{K}$. It is also $\mathcal{NP}$-complete over the complex numbers (Theorem 1 in Chapter 5 of [?]).

Example 2. Let

$$X = \{(m, x) \in \mathbb{N} \times \mathbb{C}\}$$

$$X^{yes} = \{(m, x) \in X \text{ such that } x \in \{1, 2, \cdots, m\}\}$$
with the convention that $m$ is in bit representation, while $x$ is a complex number. Hence, $\text{Size}((m, x)) = O(1 + \lceil \log_2(m) \rceil)$. Then the problem $(X, X_{\text{yes}})$ is in $NP$ over $\mathbb{C}$. The machine $M(x, g)$ can be constructing by guessing the bit decomposition $g_i$ of $x$, and computing $x - \sum g_i2^i$.

Again, $(X, X_{\text{yes}})$ is definable without constants.

3 Construction of the class $UP$

In Chapter 7 of [?], it is proved that if the problem $(X, X_{\text{yes}})$ from Example 2 would happen to belong to the class $P$, then condition (b) in Theorem 1 would be false. Therefore (b) implies $P \neq NP$ over $\mathbb{C}$.

The class $UP$ will be constructed by abstracting the same reasoning. The construction relies on some geometric properties of structured problems in $P$. The notation that follows will be used in the sequel:

Let $(X, X_{\text{yes}})$ be a structured problem with $X \in P$. We denote by $X \cap \mathbb{C}^i$ the set $\{x \in X : \text{Size}(x) = i\}$ of size $i$ instances of the problem. Then we write $\overline{X \cap \mathbb{C}^i}$ for its Zariski closure over $\mathbb{C}$. We can define a new object associated to $X$ as:

$$\overline{X} = \bigcup_{i=0,1,\ldots} \overline{X \cap \mathbb{C}^i}$$

We can think of $\overline{X}$ as the closure of $X$, indeed it is the smallest ‘closed’ problem containing $X$. Remark that in Examples 1 and 2, we have respectively $X = \overline{X}$ and $HN = \overline{HN}$.

We can also decompose each Zariski-closed set $\overline{X \cap \mathbb{C}^i}$ into a finite union of irreducible components (affine varieties). Thus it makes sense to write $\overline{X}$ as the countable union:

$$\overline{X} = \bigcup X_j$$

where each $X_j$ is an affine variety lying in some $\mathbb{C}^s$, where $s = \text{Size}(x), x \in X_j$.

We can further define:

$$X_{\text{yes}}^j = X_j \cap X_{\text{yes}}$$
$$X_{\text{no}}^j = X_j \setminus X_{\text{yes}}$$

(See Figure 1). Using this notation,
This is Problem $(X, X^{\text{yes}})$ from Example 2, restricted to the inputs $(m_0, m_1, x)$ of size 3. $X$ is represented by the four (complex!) lines and $X^{\text{yes}}$ by the dots. Each of the complex lines is irreducible, and hence corresponds to a different $X_i$.

Figure 1: $(X, X^{\text{yes}})$ from Example 2

**Definition 5.** The class $\mathcal{UP}$ is the class of all structured problems $(X, X^{\text{yes}})$ such that $X \in \mathcal{P}$ and for all $X_i$, there is a non-zero polynomial $f_i \in \mathbb{Z}[x_1, \cdots, x_{s_i}]$, where $s_i = \text{Size}(x)$ for $x \in X_i$, with the following properties:

1. $\tau(f_i)$ is polynomially bounded in $S_i$.
2. $X_i^{\text{yes}} \subseteq Z(f)$ or $X_i^{\text{no}} \subseteq Z(f)$

**Proposition 1.** $\mathcal{P} \cap \mathcal{K} \subseteq \mathcal{UP}$

*Proof of Proposition 1.* Let $(X, X^{\text{yes}})$ be in $\mathcal{P} \cap \mathcal{K}$. Let $M = M(x)$ be the machine that recognizes $x \in X^{\text{yes}}$ in polynomial time, where the input $x$ is assumed to be in $\overline{X}$. Although it is possible that an $x \in X_i$ is not in $X$, it is still possible to recognize $x \in X^{\text{yes}}$ in polynomial time. Indeed, $X$ is also in $\mathcal{P}$. The machine $M(x)$ will check $x \in X$ and $x \in X^{\text{yes}}$.

Now we apply elimination of constants (Theorem 3), and choose $M$ to be constant-free.

The nodes of the machine $M$ are supposed to be numbered. Given an input $x$, the path followed by input $x$ is the list of nodes traversed during the computation of $M(x)$.

When the input is restricted to one of the affine varieties $X_i$’s, we can define the canonical path (associated to $X_i$ as the path followed by the generic point of $X_i$). This corresponds to the following procedure:
At each decision node, at time $T$, branch depends upon an equality $F_T(x) = 0$, where $x$ is the original input. The polynomial $F$ can be computed within the machine running time. In case $F_T(x) = 0$ for all $x \in X_i$, we follow the Yes-path and say that this branching is trivial.

If not, we follow the no-path and say that this branching is non-trivial. The fact that $X_i$ is a variety is essential here, since it guarantees that only a codimension $\geq 1$ subset of inputs may eventually follow the Yes-path at this time.

The set of inputs that do NOT follow the canonical path can be described as the zero-set of

$$f_i = \prod F^T$$

where the product ranges over the non-trivial branches only. The polynomial $f_i$ can be computed in at most twice the running time of the machine $M$ restricted to $X_i$. By hypothesis, this is polynomial time in the size of $x \in X_i$.

Since we assumed that $M$ returns only 0 or 1, the set of the inputs that follow the canonical path (i.e. $Z(f_i)$) is either all in $X_i^{yes}$ or all in its complementary $X_i^{no}$.

There are now two possibilities. First possibility, $X_i^{yes}$ has measure zero in $X_i$, and therefore it must be contained in $Z(f_i)$. Second possibility, $X_i^{yes}$ has non-zero measure, hence it contains the complementary of $Z(f_i)$, and hence $X_i^{no}$ is a subset of $Z(f_i)$.

\[\square\]

### 4 Proof of the Theorems

*Proof of Theorem 1.*

(a) $\Rightarrow$ (b) is trivial, refer to [?] Chapter 7.

(b) $\Rightarrow$ (c): Let $(X, X^{yes})$ be the problem in Example 2. Since $X_i^{no}$ is generic in $X_i$, all inputs in $X_i^{yes}$ should escape the canonical path. Hence, if $f_d$ is the polynomial that defines the canonical path, $f_d(i) = 0$ for $i = 1, 2, \ldots, d$. But then it cannot be evaluated in time $\text{polylog}(d)$, by hypothesis (b). Hence, under the assumption (b), the problem $(X, X^{yes})$ is not in $\mathcal{UP}$. It does belong to $\mathcal{NP} \cap \mathcal{K}$, so $\mathcal{UP} \nsubseteq \mathcal{NP} \cap \mathcal{K}$.

(c) $\Rightarrow$ (d): Using Theorem 2, Condition (c) implies that $(HN, HN^{yes}) \notin \mathcal{UP}$. However, since $(HN, HN^{yes}) \in \mathcal{K}$, Proposition 1 implies $(HN, HN^{yes}) \notin \mathcal{P}$. Hence $\mathcal{P} \neq \mathcal{NP}$ over $\mathbb{C}$.

\[\square\]
Proof of Theorem 2. Let \((X, X_{\text{yes}}) \in \mathcal{NP} \cap \mathcal{K}\) and assume that \((HN, HN_{\text{yes}}) \in \mathcal{UP}\). We have to show that \((X, X_{\text{yes}}) \in \mathcal{UP}\).

For each \(X_i\), one can embed \((X_i, X_{\text{yes}}^i)\) into some \((HN_i, HN_{\text{yes}}^i)\) as follows:

Let \(M = M(x)\) be the deterministic polynomial time machine to recognize \(X\), and let \(N = N(x, g)\) be the non-deterministic polynomial time machine to recognize \(X_{\text{yes}}\). We can assume without loss of generality that \(M\) and \(N\) are constant-free (Theorem 3).

Let \(T\) be the maximum running time of \(M\) and \(N\) when the input is restricted to \(X_i\). Let \(\phi(x)\) be the combined Register Equations of machines \(M\) and \(N\) for time \(T\) (Theorem 2 in Chapter 3 of [?]). Thus, \(\phi(x)\) is a system of polynomial equations with integer coefficients and indeterminate coefficients \(x_1, x_2, \cdots\). The polynomial system \(\phi(x)\) can be constructed in polynomial time from \(x\), and the size of \(\phi(x)\) is polynomially bounded by the size of \(x\).

We claim that \(\phi(X_i)\) is contained in some \(HN_j\), and that in that case \(\phi(X_{\text{yes}}^i) \subseteq HN_{\text{yes}}^j\) and \(\phi(X_{\text{no}}^i) \subseteq HN_{\text{no}}^j\).

Indeed, \(X_i \subseteq \mathbb{C}^s\) for some \(s\), and \(\phi(\mathbb{C}^s) \subseteq HN_j\) for some \(j\). Then \(x \in X_i\) belongs to \(X_{\text{yes}}\) if and only if the corresponding \(\phi(x)\) has a solution over \(\mathbb{C}\).

We now distinguish two cases:

Case 1: \(HN_{\text{yes}}^j\) has measure zero in \(HN_j\). Thus \(HN_{\text{yes}}^j \subseteq Z(\hat{f}_j)\) for an easy-to-compute polynomial \(\hat{f}_j\). In that case, since \(X_{\text{yes}}^i\) gets mapped into \(HN_{\text{yes}}^j\), the composition \(f_i = \hat{f}_j \circ \phi\) gives the polynomial associated to \(X_i\).

Case 2: \(HN_{\text{no}}^j\) has measure zero in \(HN_j\). Thus \(HN_{\text{no}}^j \subseteq Z(\hat{f}_j)\) for an easy-to-compute polynomial \(\hat{f}_j\). In that case, since \(X_{\text{no}}^i\) gets mapped into \(HN_{\text{no}}^j\), \(f_i = \hat{f}_j \circ \phi\) is the polynomial associated to \(X_i\). \(\square\)

5 Ultimate Complexity

Let \((Y, Y_{\text{yes}})\) be a problem over \(\mathbb{C}\), definable without constants and with \(Y\) semi-decidable (i.e. \(Y\) is the halting set of some machine). The closure \(\overline{Y}\) is well-defined and can be written as a countable union of irreducible varieties \(Y_i\).

For any machine \(M\) to solve \((Y, Y_{\text{yes}})\), one can produce a family of polynomials \(f_i\), vanishing on the set of inputs that follow the canonical-path of
$M$ restricted to $Y_i$. As in item (2) of Definition 5, we have

$$Y_i^{\text{yes}} \subseteq Z(f_i) \text{ or } Y_i^{\text{no}} \subseteq Z(f_i)$$

Also, for each input size $s$, one has a finite number of indices $i$ corresponding to components $i Y_i \subseteq \overline{Y}$ of size-$s$ input. We can thus maximize over those indices $i$:

$$u_M(s) = \max_{i : Y_i \subseteq \mathbb{C}} \tau(f_i)$$

This invariant may be called ‘ultimate running time’, and is a lower bound (up to a constant) for the worst-case running time of $M$. As with ordinary complexity theory, one can define the ‘ultimate complexity’ class of a problem as the class of functions $u : \mathbb{N} \to \mathbb{R}$ such that $\exists M, c > 0 : \forall xu_M(x) \leq cu(x)$ and $M$ recognizes $(Y,Y^{\text{yes}})$. This provides notions such as ‘ultimate logarithmic time’ or ‘ultimate exponential time’.

In [?], a similar construction is used to obtain lower bounds for some specific decision problems. Those problems, however, had a very simple geometric structure (for each ‘input size’, $X^{\text{yes}}$ was a finite set in $\mathbb{C}$). The motivation of this paper was to extend some of the ideas therein and in Chapter 7 of [?] to non-codimension-1 problems.