ON THE NUMERICAL SOLUTION OF POLYNOMIAL SYSTEMS:

PATH FOLLOWING IN TORIC MANIFOLDS

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Example: Cassou-Nogués polynomial system

\[ p^1 = 15b^4cd^2 + 6b^4c^3 + 21b^4c^2d - 144b^2c - 8b^2c^2e - 28b^2cde - 648b^2d + 36b^2d^2e + 9b^4d^3 - 120 \]

\[ p^2 = 30b^4c^3d - 32cde^2 - 720b^5cd - 24b^2c^3e - 432b^2c^2 + 576ce - 576de + 16b^2cd^2e + 16d^2e^2 + 16c^2e^2 + 96^4c^4 + 39b^4c^2d^2 + 18b^4cd^3 - 432b^2d^2e + 24b^2d^3e - 16b^2c^2d - 240c + 5184 \]

\[ p^3 = 216b^2cd - 162b^2d^2 - 81b^2c^2 + 1008ce - 1008de + 15b^2c^2de - 15b^2c^3e - 80cde^2 + 40d^2e^2 + 40c^2e^2 + 5184 \]

\[ p^4 = 4b^2cd - 3b^2d^2 - 4b^2c^2 + 22ce - 22de + 261 \]
There are 16 non-degenerate, finite roots, 4 of which are real. An elementary substitution $B = b^2$ provides an easier system, with 8 non-degenerate, finite roots of which 4 are real.

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Root counting

Bézout’s Theorem: If $A_i = \{a : \sum a_j = d_i\}$, then the generic number of roots of $f$ is $\prod d_i$.

Kushnirenko’s Theorem: If $A_1 = \cdots = A_n$, then the generic number of roots of $f$ is $n! \text{Vol(Conv}(A_1))$.

Bernstein’s Theorem: The generic number of roots of $f$ is $n! MV(\text{Conv}(A_1), \cdots, \text{Conv}(A_n))$. 
The Minkowskii linear combination of convex sets $A_1, \cdots, A_n$ is:

$$\lambda_1 A_1 + \cdots + \lambda_n A_n = \{ \lambda_1 a_1 + \cdots + \lambda_n a_n : a_1 \in A_1, \ldots, a_n \in A_n \}$$

Example:
The Mixed volume of $A_1, \cdots, A_n$ is

$$MV(A_1, \cdots, A_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \text{Vol}(\lambda_1 A_1 + \cdots + \lambda_n A_n)$$
The maximum number of isolated roots depends only on the supports. For generic or random complex coefficients, the number of roots is exactly the maximum number of isolated roots.

Bézout’s Theorem counts the roots in complex projective space. Kushnirenko’s and Bernshtein’s theorems count roots in a toric variety.
Since the root count depends only on the supports, we can use homotopy algorithms for solving polynomial systems.

Let \((f_t)\) denote a path in coefficient-space, where:

- \(f_0\) is a system with known roots.
- \(f_1\) is a system with unknown roots.

If \(x_0\) is a known root of \(f_0\), we may extend it continuously to a root \(x_t\) of \(f_t\), unless

\[
\lim_{t \to t^*} \left\| \frac{\partial x_t}{\partial t} \right\| = \infty
\]
The complexity of path-following depends on:

- The number of roots to be followed
- The length of the path $f_t$
- The condition number
The condition number may be defined analytically as:
\[ \mu(f, x) = \| \frac{\partial x}{\partial f} \| \]

or geometrically by
\[ \mu(f, x) = \frac{1}{d(f, \Sigma_x)} \]

Both definitions are highly sensitive to the choice of the metric structure.
Mixed volume and toric varieties:

Let $A$ be a matrix with integer coefficients. The row $A^\alpha$ corresponds to the monomial

$$z_1^{A_1^\alpha} \cdots z_n^{A_n^\alpha}$$

Polynomials with support in $A$ belong to an inner-product space $(\mathcal{F}_A, \langle \cdot, \cdot \rangle)$ so that:

$$\langle f, g \rangle = \bar{f} \cdot C \cdot g^T$$

Veronese embedding maps $z$ into $V_A(z) = (z^{A_1^1} : \cdots : z^{A_M^M})$ so that

$$f(z) = 0 \iff f \cdot V_A(z) = 0$$
We will work with a variation of Veronese embedding. Let $T^n \overset{\text{def}}{=} \mathbb{C} \mod (2\pi \sqrt{-1} \mathbb{Z})$. Then, set:

$$v_A : T^n \rightarrow \mathcal{P}_M^{-1}$$

$$p + q\sqrt{-1} \rightarrow C^{-1/2}\exp(A(p + q\sqrt{-1}))$$
Expected number of roots in the mixed case:

\[
AVG = \int_{\mathcal{X}} \frac{e^{-\|f\|^2/2}}{(2\pi)^{\dim \mathcal{X}}} \# \text{roots} = \int_{(p,q) \in T^n} \int_{\mathcal{X}_{p,q}} \frac{e^{-\|f\|^2/2}}{(2\pi)^{\dim \mathcal{X}} \frac{1}{NJ}}
\]

Where:

The integral of this is a wedge of differential 2–forms.

\[
NJ^{-1} = (-1)^{(n-1)/2+1/2} \wedge f^i \cdot (Dv_{A_i})_{(p,q)} dp \wedge \cdots \wedge \bar{f}^i \cdot (Dv_{A_i})_{(p,-q)} dq
\]
In projective space $\mathbb{P}^{m-1}$, the Fubini Study metric is usually defined through the Kähler form

$$\omega \overset{\text{def}}{=} -dJ^*d\left(\frac{1}{2} \log \|z\|^2\right)$$

Given a matrix $A$, we already defined a modified Veronese embedding $v_A$. Now we can define a Kähler form on $T^n$ by pulling $\omega$ back by $v_A$.

$$\omega_A \overset{\text{def}}{=} v_A^*\omega$$
The momentum map and convex sets

Let $A \subseteq \mathbb{Z}^n$, and let $\mathcal{A} = \text{Conv}(A)$. Then there is a convex function $g_A : \mathbb{R}^n \to (\mathbb{R}^n)^\vee$ such that:

- $\mathcal{A} = \{ \nabla g_A(p) : p \in \mathbb{R}^n \}$
- $D^2 g_A = 2 Dv_A(p)^T Dv_A(p)$
- Explicit formula: $g_A = \frac{1}{2} \log v_A(p)^T v_A(p)$
- We have also: $\omega_A = -dJ^* dg_A$

The mapping $\nabla g_A : T^n \to (\mathbb{R}^n)^\vee$ is called the Momentum Map. It was introduced in the context of celestial mechanics by Smale (1970) and Souriau (1970).
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<th>$\mu_{\text{scaled}}$</th>
<th>$\mu_{\text{toric}}$</th>
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