We present a simple, surprisingly fast algorithm for deciding whether a collection of multivariate polynomials with integer coefficients has a complex root. This problem is sometimes referred to as Hilbert’s Nullstellensatz (HN). The algorithm we present is completely different from the usual Groebner basis, homotopy, or resultant techniques, and is instead based on an oracle sampling trick for checking the density of primes $p$ for which the equations have a root mod $p$. The punchline is that under a plausible number-theoretic assumption, our algorithm is sufficiently fast to yield an interesting implication relating HN to complexity theory: $\text{HN} \notin \text{P} \Rightarrow \text{P} \neq \text{NP}$.

The underlying assumption is a weakening of the Generalized Riemann Hypothesis (GRH) to a statement about the density of zeroes of the underlying Dedekind zeta function. In particular, the results above were previously known (via seminal work of Pascal Koiran) only under the assumption of the full GRH. We also present similar speed-ups for computing the dimension of a complex algebraic set and detecting rational points on certain zero-dimensional algebraic sets.